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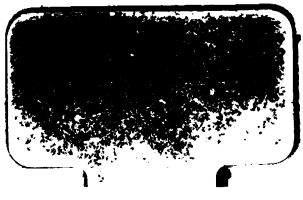

ELEMENTARY STATICS

BY THE REV.

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PREFACE.

A TREATISE on Elementary Statics for the use of Students in Elphinstone College was in course of preparation, by my lamented friend, Mr Rawlinson, when death brought all his labours in the cause of education to a sudden end. What he had written on this subject was sent to me with a request that I would prepare it for the press. In doing so I found a considerable portion of the plan of the work only sketched out in the manuscript, this I had to fill in as best I could. I am also responsible for most of the problems and illustrations and for the examination papers which appear in the following pages. I feel also that I am solely responsible for the faults that will be found in them. Whatever portion of the work has any merit may be safely reckoned as coming from the pen of my friend, whatever is weak and obscure as coming from my own.

It will perhaps be felt that some important points have been omitted which ought to have found a place in this Treatise, and that others have not been considered with sufficient fulness. The absence of any explanation of the Principle of Virtual Velocities—a principle of which, I think,

great use might be made in an Elementary Treatise on Statics—and the want of a larger collection of Problems, and of advice and guidance in their solution, are doubtless blemishes. But I have been in a measure constrained to brevity. The work is published at the charge of Her Majesty's Government, and I have not felt myself justified in exceeding, more than could be avoided, the space and expense which, in the estimate previously supplied by the Publishers, was deemed sufficient for printing the manuscript as it was left by Mr Rawlinson.

EDWARD STURGES.

KENCOTE RECTORY,
October, 1860.

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ELEMENTARY STATICS.

CHAPTER I.

PRELIMINARY NOTIONS.

1. STATICS is the science which investigates the conditions which must hold among any number of forces which keep a body at rest. We define force thus:

DEF. *Force is the cause which produces or tends to produce change in the state of rest or motion of a body.*

In Statics we are concerned only with those forces which *tend* to produce the change. When such a change is actually produced the treatment of the subject falls under another branch of Mechanics, called *Dynamics.*

2. The first thing to be done in order to bring force under Mathematical treatment is to find some means by which the *intensity* of different forces may be numerically represented.

It will be guessed from the definition of force, that all our knowledge of this subtle agent is derived from the effect we see it produce rather than from any direct knowledge of it. Thus when we lift a weight with the hand, we are conscious of a force exerted on the weight by the muscles of the arm. We know such a force can be called forth at pleasure by a connection existing between the muscles and the brain, but when we have thus traced the source of this power from one seat to another, and are asked "What *is* this force?" we can

only answer, "It is *something* which enables us to lift a certain weight."

Accordingly we *estimate* force by the effect it produces or is capable of producing.

To arrive at a *numerical estimate* of a force, we must further fix, by common consent, upon some standard force which producing a known effect may be taken as a unit of force and be represented by the figure 1, a force producing twice that effect will be represented by 2, and so on.

It is usual to take as our unit of force, that force which will sustain 1 lb. weight when acting vertically upwards, a force which would sustain 2 lbs. will then be represented by the figure 2, a force which will sustain 3 lbs. by the figure 3, and generally a force which will sustain P lbs. will be represented by P.

This then enables us to represent and compare forces numerically.

3. Force is always measured numerically upon this principle, that is, by the weight it will sustain when acting vertically upwards; although the unit of weight varies with place and circumstance. For instance, in dealing with very large forces, we might speak of a force of so many tons instead of pounds; and an inhabitant of Bombay may speak of a force of so many seers.

Now further, to clear up our ideas on this subject, let us inquire what is meant by a unit of weight; a pound for instance. What is a pound? The question seems easy enough. We have in our eye a piece of lead or iron of a certain size. We know by experience pretty well what a pound of iron is, or a pound of sugar; but we want to know what security we have for a pound being the same weight all the world over. If it were suspected that all the pound weights in India had by wear or other cause lost some of their original weight, how could we get back to our standard?

To shew some of the difficulties of fixing upon a standard of comparison, suppose, for a moment, we had agreed to say

that a pound should be a cubic inch of iron. Now there are two objections fatal to such a selection of a standard. In the first place metal expands by heat, and consequently our pound would be more than a cubic inch in hot weather, less in cold. And secondly, different kinds of iron have different densities, so that a cubic inch of one kind would weigh more or less than a cubic inch of another.

We must therefore search for a standard in some substance which is or can be made free from these objections. Now pure distilled water at a given temperature is always of the same weight per cubic inch. And accordingly a pound is fixed by Act of Parliament to be the weight of 22·185 cubic inches of pure distilled water at the temperature of 62° Fahrenheit.

22·185 cubic inches of distilled water at 60° being an invariable quantity, the pound so estimated is invariable also.

The method of measuring water with sufficient accuracy to obtain 22·185 inches, forms no part of our present inquiry. It is sufficient to say that it can be done. It will be seen that our unit of weight involves the idea of a unit of length, for we say it is so many *inches* of a certain substance.

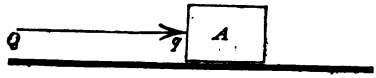
What then is an inch? Here again we must seek for some standard of comparison which neither heat nor want of uniformity in the material, nor any other cause will affect. The Act of Parliament defines an inch to be the 39·1393th part of the length of a simple pendulum vibrating seconds, in the latitude of London, at the level of the sea.

This again involves the idea of a unit of time, a second, but this is obtained directly from the motion of the Sun round the Earth.

4. To return now to our subject, let us consider what this *numerical estimate* of force which we have arrived at will do for us; that is, what information is conveyed to us when we talk of a force or pressure of 10lbs. Clearly only this, that a force of a certain intensity (namely 10lbs.) is acting *somewhere* in a body in *some direction*. And a very little thought will shew us that we must have this *somewhere*

pointed out as well as the *direction* in which the force acts, before we can form an idea of the effect of the force. In other words we must know not only the *intensity* of the force, but its *point of application* and its *direction*.

For instance, if I tell you a body A lying on the table is



acted upon by a force of 10lbs., you will not know whether this force is pressing the body down, or lifting it up, or pushing it along, or acting in any other direction. But if I say the force acts on the body at the point q, and in the direction represented by the line Qq, you can now form some conception of the character of this force, and will be prepared to reason on the effect it will produce on the body.

There are, then, three indispensable things which must be given us before we can say a force *is known*, namely:

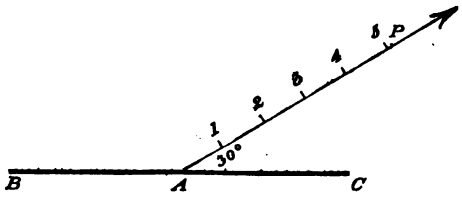
1. Its magnitude.
2. Its direction.
3. Its point of application.

Now there is a way of representing forces by which these three data may all be presented to the eye at once, namely, by straight lines.

We will explain this by an example.

Ex. Suppose a force of 5 lbs., inclined at 30° to the horizon, to be acting at the centre of a horizontal rod.

Let BC be the rod, A the centre, draw at A a straight line making an angle 30° with the horizon, take a portion of this line AP containing 5 units of length, *i.e.* as many units of length as there are units of force in the given force. Then we say AP represents the given force in every necessary particular,



In magnitude, by the number of units in its length.

In direction, by its inclination to the horizon, viz. 30°.

In point of application, viz. at A the centre of the rod.

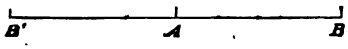
It is evident that *any* force may in this manner be fully represented by a straight line. Whenever, then, in future we say, Let such and such a straight line represent such and such a force, the student must remember that each unit of length of the line represents a unit of force, the inclination of the line to some known line gives us the direction of the force, and a given point in the line gives us its point of application.

5. The symbol P applied to a force, we have already seen, means that the force is of such intensity that it will support a weight containing P times a certain chosen unit of weight.

The mode of representing forces by straight lines, suggests to us a way in which the meaning of this symbol may be further extended.

Every one acquainted with Trigonometry is familiar with the fact that the signs + and − are used to indicate whether a line is measured in one direction or the opposite with regard to a fixed point.

Thus, if A be a fixed point in the line $B'AB$ and AB be taken equal to AB' equal to a.



If AB is called $+a$, then AB' will be represented by $-a$. We adopt a similar convention with regard to forces.

Thus if AB be taken of such a length that it represents in magnitude and direction a force P acting on a particle A, from A towards B: the symbol $-P$ will represent an equal force acting in exactly an opposite direction, geometrically represented by the line AB'. This meaning of the algebraical signs affixed to symbols representing forces must be borne in mind by the student.

6. It would be consistent with the arrangement of our subject, to give here a list of definitions of terms used in this science. But we prefer inserting them from time to time in the progress of the work, as the necessity of using the terms themselves arises. Some definitions, indeed, are included in the explanatory matter, and the student will be able to draw them thence without their being stated more explicitly. We have already defined *force*. We add here two other definitions.

DEF. A *particle*, or *material particle*, is an infinitely small portion of matter of insensible weight, incapable of division or compression.

DEF. Where two or more forces acting on a particle or system of particles, produce together no statical effect whatever, they are said to be in *Equilibrium*.

We now call the student's attention to certain truisms or axioms. These axioms, although not capable of any direct proof, are confirmed by experiment as well as by the unvarying correctness of all results based on the supposition of their truth.

AXIOM 1. *If two equal forces act upon a particle in opposite directions, that point will remain at rest.*

It is evident from this Axiom together with Axiom 3, that two equal forces acting in opposite directions at the extremities of a straight string will neutralize one another, or be in equilibrium; and the same will be true if the string instead of being straight, is passed over one or more pulleys,

assuming the absence of all friction. Hence this Axiom is sometimes spoken of as the Principle of the Pulley.

The next axiom is the converse of this.

AXIOM 2. *If a particle remain at rest, all the forces which act upon it may be reduced to two equal and opposite forces.*

AXIOM 3. *If a force be applied at any point of a system of particles rigidly connected together, its statical effect will be the same if it be removed to any other point in its direction rigidly connected with the system of points on which it acts.*

This Axiom is the Physical Principle of the transmission of Force.

The next axiom is the converse of this.

AXIOM 4. *If a force act at any point of a system of particles rigidly connected together, and being removed to any other point rigidly connected with the system, be found to produce the same statical effect, the line joining the two points is the line of action of the force.*

AXIOM 5. *If any system of forces in equilibrium be added or superimposed on another system also in equilibrium, the whole will be in equilibrium.*

This is called the principle of *superposition of forces*, and more will be said of it at a future time.

7. It is only by the diligent use of illustrations and examples that the learner can make himself familiar with that process of reasoning, upon his readiness in using which, his ultimate success in physical science must depend. We shall endeavour therefore to illustrate our subject from time to time with examples and other matters calculated to encourage this process of reasoning. This part of the work will repay the learner for a careful and close consideration.

The following is an illustration of Axiom 3.

Suppose a rope be fastened to a nail in a ceiling. Let O be the point of suspension, OC the rope.

Now if a force be acting in the direction of the rope, the axiom tells us that it will be immaterial at what point in the rope we suppose this force to act, whether at a point A or B or C. Suppose the force were a weight of 10 lbs., and we hung this at first at A with a rope of length OA, and then at B, increasing the length of rope to OB. It might be objected that the effect in the two cases would not be the same, for, in the second case we have the weight of an additional piece of rope AB. This is true, but observe, the effect *of the force* 10 *lbs.* is the same in both cases. The fact that the effect of the force at B only exceeds the effect of the force at A, by the weight of the portion of rope AB, sufficiently proves that the force of 10 lbs. does produce, as far as itself is concerned, the same effect at A and B.

In estimating the effect of forces, the student should learn to separate, when necessary, each force from its connection with all others, so as to be able to consider it separately.

Examination on Chapter I.

1. Distinguish between Statics and Dynamics.

2. What would you understand by a force 10, or a force P in the absence of other information respecting such forces?

(Ans. A force is measured numerically by the weight it would sustain if acting vertically upwards. The most common unit of weight employed in this case is 1 lb. Hence, in the absence of other information, I should understand by a force 10 or a force P, a force of such intensity that it would support 10 lbs. or P lbs. respectively, if applied to the weight in the manner stated.)

3. How is a permanent standard of measurement in length secured?

4. Define the *direction of a force.*

(Ans. If we suppose a force to act on a particle which is perfectly free to move, and not acted on by any other

force, the direction in which that particle would begin to move is the *direction of the force.*)

5. State clearly how a straight line may be made to represent the magnitude of a force.

6. If 5 represent a force of 5 lbs. pulling a body vertically upwards from the ground, what does -5 represent?

7. Enunciate the principle of the pulley.

8. Enunciate the principle of the superposition of forces.

CHAPTER II.

ON THE COMPOSITION AND RESOLUTION OF FORCES ACTING AT ONE POINT.

1. A LITTLE consideration will satisfy the student that any number of forces acting on a particle in any direction may be represented by a single force which will be equivalent to the whole system in every statical particular. For suppose such a system did not keep the particle at rest, it is evident the particle would begin to move in a certain direction. A single force of the necessary magnitude and acting in a direction exactly opposite to that in which the particle will begin to move, would keep it at rest. Call this force R, then R is in equilibrium with the whole system of forces; but if so, it is evident (Axiom 2) that the system itself is equivalent to a single force of the same magnitude as R, acting in the direction in which the particle will begin to move. R is termed *the resultant of the system of forces.*

The method of determining the resultant of different systems of forces, forms a most important branch of Statics.

This process is called the *Composition of forces.* The opposite process, of deducing from a single force an equivalent system, is termed *resolution.*

We proceed in this and the next chapter to shew how to determine the resultant in all cases where two or more forces act on a particle.

The student will notice that strictly speaking the case of two or more forces acting on a particle does not fall

into the subject "Statics" at all, unless the forces happen to be in equilibrium. But all such cases may be turned into Statical problems, by supposing such a force applied to the system as will keep the whole at rest. This force, it must be remembered, with the sign changed is the resultant of which we have been speaking.

2. *To find the resultant of two forces acting on a particle along the same line.*

If a force of P lbs. acts upon a particle in a certain direction, and another force of Q lbs. acts upon the particle in the same direction, it is evident that the two forces together will produce or tend to produce the same effect as a single force of $(P+Q)$ lbs. acting in the direction of the two forces. Or, if R be the numerical value of the equivalent force in lbs.

$$P+Q=R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If P act in one direction and Q in the opposite, and P be the greater force, we shall have

$$P-Q=R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If P be less than Q, R will be negative, which will indicate that the particle instead of being pulled forward will be pushed backward.

By observing the arrangement with regard to the algebraical signs prefixed to symbols representing forces, pointed out in Chap. I. Art. 5, the formulæ (1) and (2) will both be included in the single expression,

$$P+Q=R.$$

3. *To find the resultant of any number of forces acting on a particle along the same straight line.*

This is the preceding problem made quite general: let P_1, P_2, P_3, &c. be any number of forces acting upon a particle along the same straight line, P_1, P_2, &c. being positive or negative as the case may be, then we have

$$P_1+P_2+P_3+\ldots\ldots=R,$$

a formula which will give us a single equivalent force in every case.

Observation. It will be remarked that P_1, P_2, P_3, &c. and $-R$ are a system of forces *in equilibrium.* Therefore the above is a true statical problem. The condition of equilibrium being expressed by the equation

$$P_1 + P_2 + P_3 + \&c. - R = 0.$$

Ex. 1. Suppose a particle pulled in a certain direction by two forces of 3 and 5 lbs. respectively, and held back in the opposite direction by a force of 2 lbs.: what single force will produce the same effect on the particle as these three?

Here referring to the general formula of the preceding proposition, we have

$$P_1 = 3\ P_2 = 5\ P_3 = -2;$$
$$\therefore\ R = 3 + 5 - 2,$$
$$= 6,$$

or, the single equivalent force will be one of 6 lbs., acting in the same direction as the forces of 3 and 5.

Ex. 2. Find the condition that the forces $P_1, P_2 \ldots P_n$ may be in equilibrium.

In this case R must equal 0, therefore the required condition is

$$P_1 + P_2 + \&c. + P_n = 0.$$

Observation. This last example is a type of a large class of Statical Problems. Whenever we have an equation, or a number of equations, from which we may determine the resultant of a system of forces, by supposing that resultant to be 0, every such equation will give us a condition which must hold among the forces in order that there may be equilibrium.

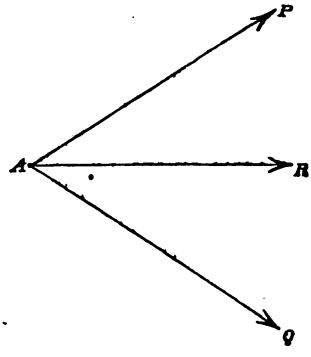
4. Hitherto we have been concerned with forces acting along the same straight line. The next proposition, in order, would be to find the resultant of two forces acting on a particle in any directions. This is in fact the great pro-

position upon which the whole science of Statics is founded. Before we proceed to it we have some preliminary matter to present to the student.

5. It has already been shewn, in Art. 1, of this Chapter, that any number of forces acting on a particle must have a resultant; this perhaps will appear plainer in the following manner.

Let P and Q be two forces acting in the particle A in different directions. Draw AP, AQ representing the forces in magnitude and direction.

Now consider what tendency to motion these two forces produce on A. We observe



1st. That the effect of the force P is to make A tend to move in direction AP.

2ndly. That the effect of the force Q is to make A tend to move in direction AQ.

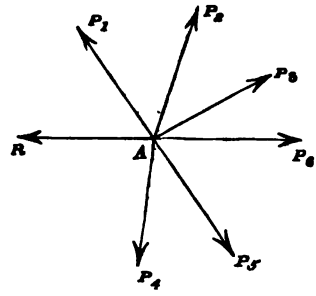
3rdly. Both these forces act at once, but it is evident the particle cannot move in both directions at once. Hereafter we shall be able to ascertain both, the direction in which A would move under these circumstances, and also the effect in magnitude of the two forces upon A. At present we only wish to point out that A must evidently have a tendency to move in some direction lying between AP and AQ, AR suppose, therefore a single force, of a certain magnitude, acting in direction AR, would produce the same effect on A as P and Q together. Such a force is the resultant of P and Q.

6. In the same way it may be shewn that if any number of forces act on a point, these forces together produce the same effect as a single force R suppose. For if $P_1, P_2 \ldots P_n$ be n forces acting upon a point A, any two of them as P_1, P_2 may

be removed and replaced by a single force R_1; similarly R_1 and P_3 may be removed and replaced by a single force R_2 and so on, until there remains only a single force R.

R is said to be the resultant of $P_1, P_2 \dots P_n$.

7. It follows immediately from the foregoing that *if any number of forces act upon a particle and keep it at rest, any one of them is equal and opposite to the resultant of all the rest.*



Let $P_1, P_2 \dots P_6$ be six forces acting on a point A. Then any one of them as P_6 is equal and opposite to the resultant of all the rest.

By Art. 6, it is evident that $P_1, P_2 \dots P_5$ may be removed and replaced by a single force R, acting in some direction AR suppose. Let this be done. Then we have two forces acting in directions AP_6 and AR on the particle A and keeping it at rest. Hence it follows that P_6 and R_1 must be equal and opposite.

Therefore any one force as P_6 is equal and opposite to the resultant of all the other forces $P_1, P_2 \dots P_5$.

8. We may now proceed to determine the *direction* of the resultant in the particular case of two equal forces P and P acting upon a particle A, in different directions.

The consideration of this case will shew us that as A can have no more tendency to move in the direction of one force than that of the other, inasmuch as the forces are equal, there can be no reason why the direction of the resultant should be more inclined to one force than to the other. Therefore the direction of the resultant will bisect the angle between the forces.

It will be noticed here that the direction of the resultant is that of the diagonal of the parallelogram, constructed upon the lines representing the two forces.

9. The proposition which enables us to determine the direction and magnitude of the resultant of *any two* forces acting on a particle in different directions is called the "Parallelogram of Forces."

The student must thoroughly understand the principles involved in this important proposition before he can make any progress in the science.

We proceed now to enunciate it.

The Parallelogram of Forces.

If two forces acting upon a particle in different directions be represented in magnitude and direction by two straight lines drawn from the particle, then the diagonal of the parallelogram constructed upon these two straight lines will represent the resultant of the forces in magnitude and direction.

We postpone the proof of this proposition for the present.

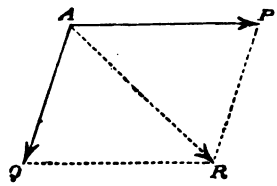
The following illustration will make the meaning of the enunciation quite plain.

Let P and Q be two forces acting on the particle A in directions AP, AQ respectively.

Take AP, AQ, each as many units of length as there are units in the forces P and Q respectively,

i. e. if P be 10 lbs.

and Q be 7 lbs.



and one inch be chosen as our unit of length,

take $AP = 10$ inches,

$AQ = 7$ inches,

then the lines AP, AQ represent the forces P and Q in

1st. Magnitude.

2nd. Direction.

3rd. Point of application.

Complete the parallelogram $APRQ$, and draw the diagonal AR.

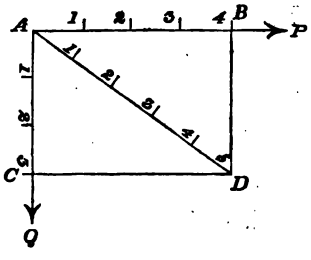
Then the proposition states that the diagonal AR will represent the resultant of P and Q in the above three particulars, viz. in magnitude, in direction, and evidently also in point of application, P and Q might therefore be removed, and the single force, represented by the line AR, substituted in their stead.

10. Now assuming the truth of the "parallelogram of forces," we give here some examples which will serve to make the student familiar with its use and principle.

Ex. 1. Two forces of 4 and 3lbs. respectively, act on a particle at right angles to one another. Find the direction and magnitude of their resultant.

Let A be the particle,

AP, AQ the direction of the forces.



Let $\frac{1}{10}$th of an inch be our unit of measure.

Take AB $\frac{4}{10}$ths,

AC $\frac{3}{10}$ths of an inch.

Then AB, AC represent the forces respectively.

Complete the parallelogram $ABDC$. Draw AD.

Then, by our assumption, AD represents the resultant in magnitude and direction.

If AD be measured it will be found to be $\frac{5}{10}$th of an inch.

And if the angle BAD be measured it will be found to be about $36^\circ . 51'$.

Therefore, the resultant is a force of 5lbs. making an angle of $36^\circ . 51'$ with the larger force.

This is solving the problem by actual admeasurement.

The result would be obtained analytically thus:

$$\begin{aligned} AD^2 &= AB^2 + BD^2 \\ &= 4^2 + 3^2 \\ &= 25; \\ \therefore\ AD &= 5. \end{aligned}$$

And sine of angle $BAD = \frac{BD}{AD}$

$$\begin{aligned} &= \frac{3}{5} \\ &= \cdot 6, \end{aligned}$$

which on reference to the table of natural sines we find to be $36^\circ . 51'$.

Ex. 2. Two forces of 5 and 10lbs. respectively, whose directions are inclined at an angle of 120° act upon a particle; find the direction and magnitude of their resultant.

Let A be the particle.

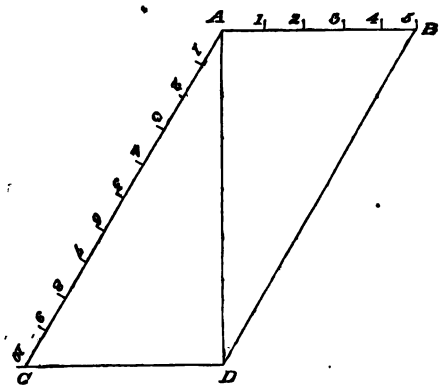
Make the angle $BAC = 120^\circ$.

Take AB containing 5 units of length,

..... AC ..... 10 .....

Complete the parallelogram $ACDB$.

Then by actual measurement we shall find



$$AD = 8{\cdot}65\ldots \text{ units of length,}$$

$$\text{angle } BAD = 90^0,$$

or the resultant will be a force of 8·65 ... lbs. making an angle 90^0 with the force 5.

Or the problem may be solved geometrically by Euclid, Book II. Prop. 13,

or by trigonometry thus,

$$AD^2 = AB^2 + BD^2 - 2AB\,.\,BD\,.\cos ABD,$$

$$\cos ABD = \cos(180 - 120) = \cos 60^\circ = \frac{1}{2};$$

$$\therefore\ AD^2 = 5^2 + 10^2 - 2\,.\,5\,.\,10\,.\frac{1}{2}$$

$$= 125 - 50$$

$$= 75;$$

$$\therefore\ AD = 5\sqrt{3} = 8{\cdot}65 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again,

$$\frac{\sin BAD}{\sin ABD} = \frac{BD}{AD} = \frac{10}{5\sqrt{3}} = \frac{2}{\sqrt{3}},$$

$$\text{and } \sin ABD = \sin 120^\circ = \frac{\sqrt{3}}{2};$$

$$\therefore \sin BAD = 1 = \sin 90^\circ;$$

$$\therefore BAD = 90^\circ \text{ ..... } (2).$$

(1) gives us the magnitude,

(2) the direction of the resultant.

11. Referring back to the preceding figure, we have the well-known trigonometrical formula which we have already used,

$$AD^2 = AB^2 + BD^2 - 2AB \,.\, BD \cos ABD.$$

Therefore if P and Q be two forces acting on a particle and inclined to one another at an angle α, and if R be the resultant, θ the angle it makes with P, the above formula becomes

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha;$$

we have also

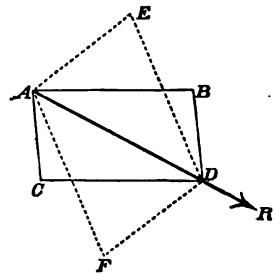
$$\sin \theta = \frac{Q}{R} \sin \alpha;$$

two general expressions which will determine the magnitude and direction of the resultant in all problems of this class.

12. The preceding examples relate to the "Composition of forces." Assuming for the present the truth of the parallelogram of forces, we proceed now to consider the converse case, namely, the "resolution" of a single force into two others.

Let R be a force acting upon the particle A, represented in direction by the line AR. Take AD of such length that it represents this force in magnitude.

Draw any line AB of any length, join BD. Complete the parallelogram $ABDC$.



Then it is evident that AD is the resultant of the two forces which are represented in magnitude and direction by the lines AB, AC, respectively.

The single force R may, therefore, be removed, and these two substituted in their stead.

The student will observe that while a single pair of forces must, by the parallelogram of forces, have one and only one resultant, in the converse case a single force may be resolved into an infinite number of pairs of forces, any one of which will be equivalent to the single force.

Thus in the preceding figure, we may draw any other line instead of AB, as AE, of any length, join ED, complete the parallelogram $AEDF$. Then, it is evident, AD is the resultant of the forces represented by the lines AE, AF; and so on, we may shew that there are an infinite number of pairs of forces, any one of which has the force R for a resultant.

In the two new forces, P and Q suppose, into which we propose to resolve any given force R acting on a particle, there are in fact four elements concerned, viz. the magnitude and direction of P, and the magnitude and direction of Q. Any two of these may be chosen arbitrarily, except in the case where we choose the two magnitudes, when it is evident our choice will be subject to the condition that the magnitudes of P and Q must be such that any two of the forces P, Q, R are together greater than the third.

Hence, four classes of problems arise from this case of the resolution of a force. Three of these are given in Example 6, in the examination paper at the end of this chapter, the fourth is treated in the following article.

13. A force R acts on a particle A, it is required to resolve it into two others P and Q, which shall act in given directions.

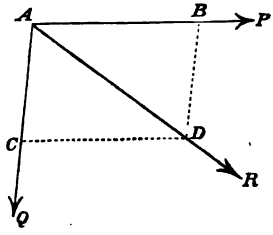
Let AR be the direction of the force R.

Take AD representing the force R in magnitude.

Through A draw AP, AQ parallel to the given directions of P and Q respectively.

Draw DB, DC parallel to AP, AQ respectively.

Then AB, AC will evidently represent the required forces in magnitude and direction.



The following is a form of the above proposition which is constantly occurring in statical problems.

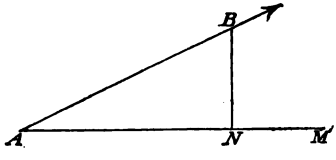
To find the effect of a force in a given direction.

Let ABP, in preceding figure, be drawn from the point of application parallel to the required direction. Then we want to find what tendency the force R has to pull the particle along the line ABP.

From D draw DB perpendicular to ABP, and complete the parallelogram $ABDC$.

Then the forces represented by the lines AB, AC, are equivalent to the force R.

But the force AC being perpendicular to AB can produce no effect at all in that direction.



Therefore the force AB represents the whole effect of R in a direction parallel to ABP.

In all future investigations, then, if AB represent a force in magnitude and direction, and we want to find the effect of that force along any line parallel to AM, draw BN perpendicular to AM, and AN will represent the portion of the force required.

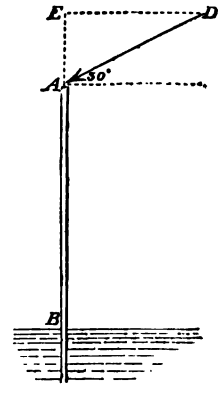
14. The following are examples of the application of the principle explained in the preceding article.

Ex. 1. AB is a stick fastened perpendicularly into the ground, DA is a force pushing on the top of the stick, whose magnitude is 5 lbs. and whose direction makes an angle of 30° with the horizon. Find the tendency of this force to push the stick on one side, and also its tendency to push the stick into the ground.

Take DA representing the force in magnitude and direction.

Produce length of stick in direction AE.

Draw DE perpendicular to AE.



Then DE will evidently represent the magnitude of the force which pushes the stick on one side.

And EA will represent the magnitude of the force which pushes it into the ground.

And since

$$DA = 5, \text{ and angle } DAE = 60^\circ,$$

$$DE = DA \sin DAE = 5 \sin 60 = 4{\cdot}33 \ldots$$

and $$EA = DA \cos 60^\circ = 2{\cdot}5.$$

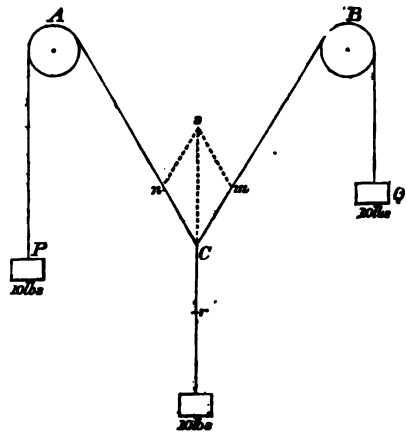
Hence the single force of 5 lbs. acting at an angle of 30° to the horizon is equivalent to two others, one pushing the stick downwards with a force of 2·5 lbs., the other on one side with a force of 4·33.... lbs.

In the following example the pulleys are supposed to require no appreciable force to make them run round, the effect of them will, therefore, be to transmit the force exerted by the

weights along the whole length of the string without alteration. This must always be understood to be our meaning when we speak of pulleys as *smooth*.

Ex. 2. *A* and *B* are smooth pulleys over which passes a string *PACBQ*. At each extremity of the string a weight of 10 lbs. is attached, and at the point *C* another weight of 10 lbs. is attached by a string. Find the position the string will take when the system is in equilibrium.

Take *Cr*, *Cm*, *Cn*, each representing a force of 10 lbs.



It is evident that at *C* we have three forces acting, one vertically downwards, and one along each of the directions *CB*, *CA*.

These forces are represented in magnitude and direction by the lines *Cr*, *Cm*, *Cn* respectively.

Complete the parallelogram *CnSm*.

Draw the diagonal *CS*.

Then *CS* represents in magnitude and direction the resultant of the forces represented by the lines *Cm* and *Cn*.

But (Art. 7) this resultant must be equal and opposite

to the force represented by the line Cr. Therefore CS is vertical, and equal 10 lbs.

And the triangle CnS is equilateral, each of its sides being equal to 10.

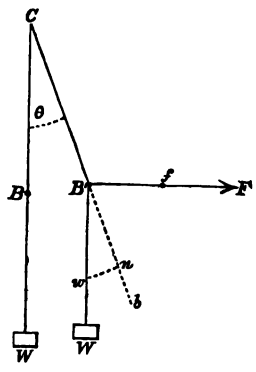
Therefore the angle $SCn = 60°$,

and similarly the angle $SCm = 60°$.

Hence the portions of the string CA, CB make equal angles with the vertical and the whole angle $ACB = 120°$, which determines the position of equilibrium.

15. When a force is conducted along a string, as in the preceding problem, the force so communicated is spoken of as *the tension* of the string. When the string is without weight, flexible and inextensible, the tension is the same at every point of its length. This is a matter of experiment. It is also evident from the fact that such a string is incapable of exerting any force in a direction perpendicular to its length, so that the whole force will continue to be transmitted without alteration along the string.

16. EXAMPLE. CBW is a string hanging from a peg C to which it is fastened, and supporting a weight W. Another string fastened at B draws the string with a force F, acting horizontally, out of the vertical, into the position $CB'W$; find the position of equilibrium.



Let the figure represent the system in a position of equilibrium. Let the angle BCB' be θ. If we determine θ the position of equilibrium will be known. The point B is kept in equilibrium by three forces F, W and the tension of the string along BC. Produce CB' in direction $B'b$.

Take $B'w$ representing W in magnitude.

Draw wn perpendicular to $CB'b$.

Then wn, $B'n$ are the resolved parts of W perpendicular to and along $CB'b$ respectively.

And $wn = wB' \sin wB'n = W \sin \theta$.

And similarly the resolved part of F perpendicular to $CB'b$ (since $WB'F$ is a right angle)

$$= F \cos \theta.$$

And these forces perpendicular to $CB'b$ must be equal to one another since they can be balanced by no part of the tension of the string $B'C$.

$$\therefore\ W \sin \theta = F \cos \theta,$$

$$\therefore\ \tan \theta = \frac{F}{W},$$

which determines θ, and the position of equilibrium is known.

The student will remark that the tension along the string $WB'C$ is not of course the same all along its length; for the effect is the same as it would be if CB' and $B'W$ were different strings.

If T be the tension of the string at B' along $B'C$, T must evidently equal the sum of the resolved parts of W and F along $B'b$.

$$\text{i.e. } W \cos \theta + F \sin \theta = T,$$

and from above $$W \sin \theta - F \cos \theta = 0.$$

Squaring these equations and adding

$$W^2 + F^2 = T^2;$$

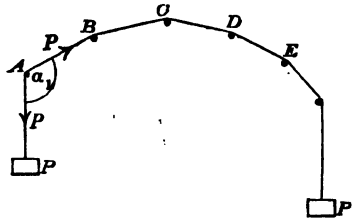
$$\therefore\ T = \sqrt{W^2 + F^2},$$

which determines the tension of the string $B'C$.

17. The following is an example of the use of the formula given in Art. 11.

EXAMPLE. A string has a weight P attached to each end, and is then laid on a number of smooth pegs $ABCD$.... Find the pressure on each peg.

The string at the point A will be kept at rest by the tension of the string AP equal to P, and also by the tension produced by the weight at the other end of the string, acting from A to B, and also equal to P, since it will be transmitted unaltered over all the smooth pulleys.



The pressure on the peg A will be the resultant of these two equal forces acting at that point, hence if R_1 be the pressure at A and angle $PAB = \alpha_1$, we have by Art. 11,

$$R_1^2 = P^2 + P^2 + 2PP \,.\, \cos \alpha_1$$

$$= 2P^2 (1 + \cos \alpha_1)$$

$$= 4P^2 \,.\, \cos^2 \frac{\alpha_1}{2};$$

$$\therefore R_1 = 2P \,.\, \cos \frac{\alpha_1}{2}.$$

Similarly if R_2 be the pressure on peg B, and angle $ABC = \alpha_2$,

$$R_2 = 2P \,.\, \cos \frac{\alpha_2}{2};$$

and similarly for all the other pressures.

Examination on Chapter II.

1. Define the term "resultant" of forces.

2. "To find the resultant of two forces acting upon a particle and not in the same straight line." By what name is this Proposition known, and how is it usually enunciated?

(N.B. *In the following examples the parallelogram of forces is to be assumed as true.*)

3. Two equal forces act upon a particle, and the angle between their directions is 60°; find the magnitude and direction of their resultant.

4. Two forces represented by 2 and 3 are inclined to each other at angle 45°; find the magnitude and direction of their resultant.

5. At what angle must two forces 3, 4 act so that their resultant may be 5?

6. A force of which the direction and magnitude is known acts upon a particle. Resolve it into two others and determine the remaining particulars of each of the component forces in the following cases:—

(1) When one of them is given in magnitude and direction.

(2) When one is given in magnitude and the other in direction.

(3) When both are given in magnitude.

7. Three forces represented by 2, 5 and 9 act on a particle. Is it possible for them to be in equilibrium?

8. Shew that the tendency of a force in any direction is found by multiplying the force by the cosine of the angle between that direction and the direction of the force.

9. Two strings PA, QA, are tied to a point A and drawn with a force 3 and 4 respectively: find what the angle PAQ must be in order that the strain on A may be 6.

10. The resultant of two forces is 10 lbs., one of them is equal to 8 lbs., and the direction of the other is inclined to the resultant at an angle of 36°; find the other force, and the angle between the two.

11. A and B are smooth pulleys; a string $PACBQ$ passes over them. At a point C in the string a weight of 10 lbs. is attached; find what weights must be attached to P and Q respectively that the angle ACB may be 120° when the system is in equilibrium.

12. In example, Art. 16, find W and T by the triangle of forces, without the aid of Trigonometry.

CHAPTER III.

ON THE COMPOSITION AND RESOLUTION OF FORCES ACTING ON A POINT.

1. THE student will now be sufficiently familiar with the method of representing forces by straight lines, and reasoning thereon, to follow the *proof* of the *parallelogram of forces.*

The principle on which this proof is made to depend is that laid down in Axiom 3 (Chapter I. Art. 6), viz. that "if a force be applied at any point of a system of particles rigidly connected together, its statical effect will be the same if it be removed to any other point of its direction rigidly connected with the system on which it acts," and the converse of this as stated in Axiom 4.

We now proceed to enunciate and prove the proposition.

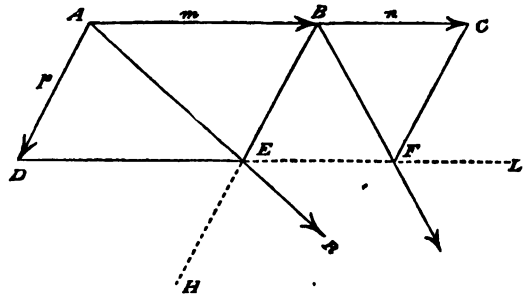
PROP. *If two forces acting upon a particle in different directions be represented in magnitude and direction by two straight lines drawn from the particle, then the diagonal of the parallelogram constructed upon these two straight lines will represent the resultant of the forces in magnitude and direction.*

1st. As far as regards the *direction* of the resultant.

The proposition is evidently true for two equal forces p and p as far as regards the direction of the resultant (Chap. II. Art. 8).

Assume it so far true for two unequal forces p and m, and also for two p and n, we shall shew it will, on this assumption, be necessarily true for the two p and $m+n$.

Let A be the particle on which the forces act. Draw AB,



AD representing the forces m and p respectively in magnitude and direction.

Produce AB, and make BC proportional to the force n in the same ratio as AB is to force m.

Then since a force may be transferred to any point of its direction, BC will represent the force n in magnitude and direction.

Complete the parallelogram $ABED$, and draw the diagonal AE.

Then, by hypothesis, AE represents the resultant (R suppose) of p and m in *direction*.

And, by axiom, the point of application of this resultant may be supposed to be at E, a point in its direction rigidly connected with A.

Now since the force R when acting at A is equivalent to the two p and m, when at E it may evidently be resolved back again to the same forces acting parallel to their original directions.

Produce DE to L, BE to H.

Then instead of the forces p and m at A,

we have a force m acting along EL at E,

and a force p acting along EH at E.

We may again change the points of application of these two forces so that

m be supposed to act at F

and p at B.

Complete the parallelogram $EBCF$ and draw the diagonal BF.

Then, by hypothesis, BF represents in direction the resultant of the two forces p and n acting at B.

And this resultant may be supposed to act at F a point in its direction, instead of A.

At F it may be resolved again into the forces p and n. We have now removed the forces p and $m+n$ which acted at A to F.

Therefore F must be a point in their resultant.

Whence the direction of the resultant of p and $m+n$ is along the diagonal of the parallelogram constructed on the lines representing them.

Now, we know the proposition to be true for the direction of the resultant of two equal forces p and p.

Since therefore it is true for p and p, and p and p, by the preceding it is true for p and $p+p$ or $2p$.

And since it is true for p and p, and p and $2p$,

it is true for p and $2p+p$ or $3p$,

and so on; therefore generally it is true for

p and rp, where r is a whole number.

And again, since it is true for rp and p, and rp and p,

it is true for rp and $2p$,

and so on by similar successive deductions it may be proved to be true generally for

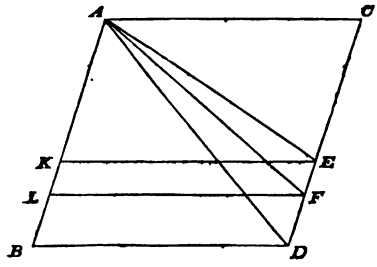
rp and sp, where s is a whole number;

therefore it is true for all commensurable forces, i.e. for all forces the ratio of whose magnitudes can be expressed by the ratio of two whole numbers.

This proves the proposition, as far as regards the *direction* of the resultant.

Note. That this proof extends also to incommensurable forces (such as the forces $\sqrt{3}$ and 2, whose ratio cannot be represented by two whole numbers) may be shewn in this way:

Let AB, AC represent in magnitude and direction two of the commensurable forces.



Complete the parallelogram $ACDB$. Draw AD.

Then we have to shew that AD represents in *direction* the resultant.

If it do not, let any other line as AE represent that direction.

Divide AC into a number of equal parts, such that each is less than ED.

Mark off along CD parts each equal to one of these.

Then one division will evidently fall between E and D, at F suppose.

Join AF, and draw EK, FL parallel to AC.

The forces represented by AC, AL are commensurable.

Therefore, their resultant acts along AF (by Prop.),

i.e. the resultant of two forces AC, AL lies further from AC than the resultant of the two AC, AB, AB being greater than AL, which is manifestly absurd.

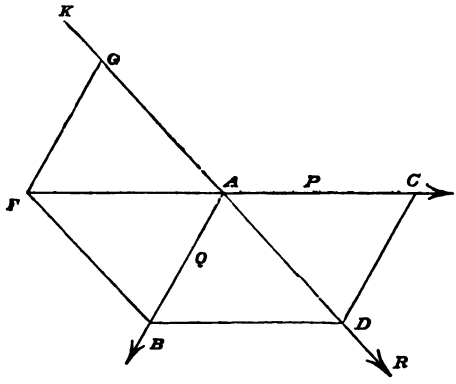
Therefore AE is not the direction of the resultant of AC, AB.

In the same manner it may be proved that no other direction but AD can be that of the resultant of those incommensurable forces.

Whence our proposition is true for *all* forces as far as the *direction* of the resultant is concerned.

2ndly. As regards the *magnitude* of the resultant.

Let AC, AB represent in magnitude and direction two forces P and Q.



Complete the parallelogram BC.

Then, by what has already been proved, the diagonal AD will be the direction of the resultant of P and Q.

Produce AD backwards towards K.

Then if we suppose a force equal in magnitude to the resultant of P and Q to act along AK, this force and P and Q will be in equilibrium.

But if any number of forces be in equilibrium any one is equal and opposite to the resultant of all the rest.

Therefore the force P must be equal and opposite to the resultant of the force acting along AK and Q.

Produce CA to F, and make AF equal to AC.

Then AF represents in magnitude as well as direction the resultant of Q, and the force along AK.

Join FB, and draw FG parallel to AB.

Then since FAC is parallel to BD,

and $FA = AC = BD$;

therefore also FB is equal and parallel to AD.

Now AF represents in magnitude and direction the resultant of Q and the force acting along AK.

But the forces represented in magnitude and direction by the lines AG, AB, have a resultant acting along AF.

Therefore AG must represent in magnitude the force which we supposed to act along AK. But this force is equal and opposite to the resultant of P and Q.

Therefore the magnitude of the resultant of P and Q

$$= AG = FB = AD.$$

That is, the magnitude of the resultant of P and Q is represented by the diagonal of the parallelogram, of which the sides represent the magnitude of the forces P and Q.

Therefore, if two forces, &c. Q.E.D.

We have now established the truth of the parallelogram of forces upon evidence of the same kind as that on which the Theorems of Euclid rest.

2. The parallelogram of forces is sometimes stated in a form in which it is known as the *triangle of forces*. We proceed to enunciate it.

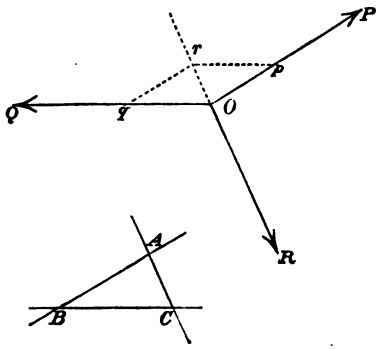
Observation. The student must remember that all the forces with which we are concerned are supposed throughout this treatise to act in one plane, and that plane to be coincident with the plane of the paper. For brevity, we shall in the next and all other investigations leave this to be understood. Of course all our operations are supposed to be carried on in the same plane.

PROP. *If three forces acting upon a particle keep it at rest, and if three lines be drawn parallel respectively to the directions of the forces, the sides of the triangle so formed will be proportional to the magnitudes of the forces.*

Let O be the particle,

OP, OQ, OR the directions of three forces P, Q, and R, acting upon it and keeping it at rest.

Take Op, Oq representing P and Q respectively in magnitude.



Complete the parallelogram $Oprq$.

Join Or, then, by the parallelogram of forces, Or is in the same straight line with OR.

And the lines Op (or rq), Oq, Or are proportional to the magnitudes of P, Q, R respectively.

Now draw three straight lines AB, BC, AC parallel to the directions of P, Q, R respectively, that is, parallel to rq, Oq, Or, then the triangle ABC so formed is evidently similar to the triangle Orq;

$$\therefore AB : BC : AC = rq : Oq : Or$$
$$= P : Q : R.$$

Therefore, if three forces, &c. Q. E. D.

3. The converse of this proposition we enunciate as follows:

PROP. *If the sides of a triangle taken in order represent in magnitude and direction three forces acting on a particle, those three forces are in equilibrium.*

Observation. By *taken in order* we mean that if AC (see preceding figure) represent a force acting from A towards C, the other two forces must be supposed to act from C towards B, and B towards A respectively; this is expressed by reading the sides, AC, CB, BA, *not* AC, AB, BC or AC, CB, AB. In neither of which last two cases would the proposition be true.

This proposition may be easily proved as follows:

Let O be the particle, draw OP, OR, OQ parallel respectively to BA, AC, CB.

Take $Op = BA$, $Oq = CB$.

Complete the figure as before.

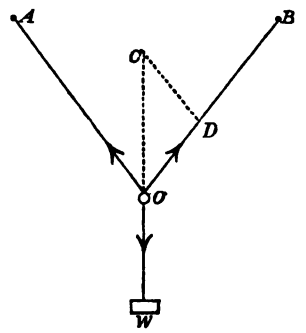
Then Or must be in the same straight line with OR.

And the triangles ABC, rOq are similar and equal, but the forces represented by the sides of the triangle rOq are in equilibrium by the parallelogram of forces.

Therefore, if the sides of a triangle, &c. Q. E. D.

PROBLEM. The following is an example of the use of the triangle of forces.

A string AOB, fastened at A and B, passes through a smooth ring O to which a weight W is attached; shew that in the position of equilibrium the vertical line through O will bisect the angle AOB.



We have here three forces acting at the point O—the weight, the tension of the string along OB, and the tension along OA.

Produce WO to any point C.

And at C draw CD parallel to OA.

Then the sides of the triangle COD will, by the triangle of forces, be proportional to the magnitudes of the three forces acting at O; i.e. CO to the weight, OD to the tension along OB, DC to the tension along DC.

But the tension along OA must equal tension along OB.

(Since AOB is the same string and the tension of every part of it is therefore the same);

$$\therefore\ OD = DC;$$

$$\therefore \text{ angle } OCD = \text{angle } DOC,$$

$$\text{but angle } OCD = \text{angle } COA;$$

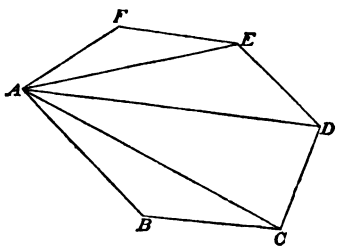
$$\therefore \text{ angle } DOC = \text{angle } AOC.$$

i.e. the two *limbs* of the string will be equally inclined to the vertical.

4. The triangle of forces is extended further in a proposition known as the *polygon of forces*, which we proceed to give.

PROP. *If the sides of a polygon taken in order are parallel in direction and proportional in magnitude respectively to a number of forces acting upon a particle; then that system of forces will keep the particle at rest.*

Let $ABCDEF$ be a polygon whose sides AB, BC, CD...



represent respectively in direction and magnitude a number of forces acting on a particle.

Join AC, AD, AE.

Then the forces represented by the lines AB, BC, CA, are by the triangle of forces in equilibrium. Therefore the third side AC *not taken in the same order* will represent the resultant of AB, AC. Hence we may remove AB, BC, and substitute a force represented by AC.

Similarly AD represents the resultant of AC, CD;

i.e. of AB, BC, CD;

and similarly AE represents the resultant of AD, DE;

i.e. of AB, BC, CD, DE.

And we have remaining three forces represented by the sides of the triangle AEF taken in order.

And a similar proof would hold if the polygon had any other number of sides.

Therefore, if the sides, &c. Q.E.D.

COR. It is evident in the above proposition that any one side taken in the opposite order to the rest will represent the force which is the resultant of all the rest.

5. The converse proposition, viz. *If any number of forces acting upon a particle keep it at rest, and a series of lines be drawn parallel in direction and proportional in magnitude to these forces respectively, the lines so drawn will form a polygon*, is easily demonstrated; and is left for the student to work out in question 4 of the examination at the end of this chapter.

6. By means of the parallelogram of forces we may determine the resultant of any number of forces ($P_1 P_2 P_3 \ldots$ suppose) acting upon a point. For by taking lines to represent these forces in magnitude and direction, we may determine the resultant (R_1) of any two of them ($P_1 P_2$), and similarly we may determine the resultant of R_1 and another of the forces P_3 and so on, until we have reduced them all to a single force R, of which the direction and magnitude will be known from the line representing it.

Or to state this in another shape; from the parallelogram of forces we may find the *Conditions of Equilibrium* of any number of forces acting on a particle. For suppose there are

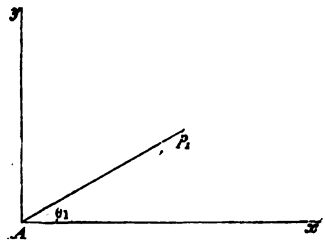
any n forces $P_1 P_2 P_3 \ldots P_n$, then determining the resultant R of $(n-1)$ of them, $P_1 P_2 \ldots P_{n-1}$, the straight line representing R must be equal in length and opposite in direction to that representing the remaining force P_n. In other words, R must be of the *same magnitude* as P_n, and be such that the algebraical sum of R and $P_n = 0$. These are the two conditions of equilibrium.

The two following propositions enable us, by the use of Trigonometry, to place this extension of the parallelogram of forces to the general problem of forces acting on a particle in a more useful shape.

The principle upon which these propositions rest, is that if a force (see Art. 13, Chap. II.) be resolved in two directions at right angles to one another, neither of these resolved parts can have any tendency to draw the particle in the direction of the other. These resolved parts represent the *whole of the force* in each of those directions respectively, and in the case of equilibrium *each of these* resolved parts must be equal to O.

7. Prop. *To find the magnitude and direction of the resultant of any number of forces acting upon a point.*

Let $P_1 P_2 P_3 \ldots$ be numerical quantities representing the magnitudes of the forces respectively.



Draw any two straight lines Ax, Ay at right angles to one another.

Through A draw Ap_1 parallel to the direction of one of the forces P_1. Take Ap_1 proportional to P_1.

Let θ_1 be the angle the direction of P_1 makes with Ax.

Then the resolved part of the force P_1 in the direction Ax is evidently $= Ap_1 \cos\theta_1 = P_1 \cos\theta_1$.

And the resolved part in the direction Aq

$$= Ap_1 \sin\theta_1 = P_1 \sin\theta_1.$$

In like manner the force P_2, whose direction makes an angle θ_2 with Ax, is equivalent to

$P_2 \cos\theta_2$ acting in the direction Ax,

$P_2 \sin\theta_2$ ..... Ay.

And so on for any number of forces.

Hence, adding together all the forces acting in the same direction, we find the forces $P_1 P_2 \ldots$ are equivalent to

$P_1 \cos\theta_1 + P_2 \cos\theta_2 + \ldots$ acting in the direction Ax,

and $P_1 \sin\theta_1 + P_2 \sin\theta_2 + \ldots$ acting in the direction Ay.

For shortness' sake let

$$P_1 \cos\theta_1 + P_2 \cos\theta_2 + \ldots = X,$$
$$P_1 \sin\theta_1 + P_2 \sin\theta_2 + \ldots = Y.$$

Now let R represent the magnitude of the resultant of the forces $P_1 P_2 \ldots$ and θ be the angle it makes with Ax; then

$R \cos\theta$ will be the resolved part of R in the direction Ax,

$R \sin\theta$ ..... Ay.

And since X and $R \cos\theta$ both represent the *whole amount of force* acting in direction Ax, we have

$$R \cos\theta = X,$$

and similarly

$$R \sin\theta = Y.$$

Therefore $$\tan\theta = \frac{Y}{X},$$

and $$R^2 = X^2 + Y^2.$$

Two expressions which determine the direction and magnitude of R respectively.

8. PROP. *To find the conditions of equilibrium of any number of forces acting on a particle.*

Suppose the forces to be all reduced to one (R), as in the preceding proposition, then in the case of equilibrium $R = 0$, or

$$X^2 + Y^2 = 0;$$

an equation which cannot be true unless

$$X = 0 \text{ and } Y = 0,$$

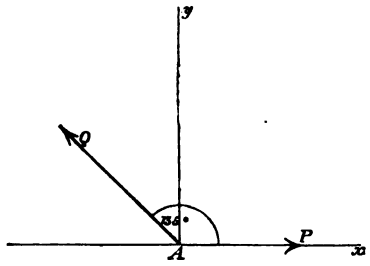
or

$$P_1 \cos\theta_1 + P_2 \cos\theta_2 + \ldots = 0,$$

$$P_1 \sin\theta_1 + P_2 \sin\theta_2 + \ldots = 0;$$

two conditions of equilibrium, which may be expressed by saying that the *sum of the forces resolved in any two directions at right angles to one another must each equal nothing.*

9. We now give some problems solved by the method indicated in the two preceding articles:



PROB. I. *Two forces acting upon a particle are inclined to one another at an angle* 135°; *find the ratio between them when the magnitude of the resultant is equal to that of the smaller force.*

Let P and Q be the two forces of which P is the less. Let A be the particle, Ax the direction of the force P.

Through A draw Ay perpendicular to Ax.

Then if θ be the angle the direction of the resultant makes with Ax, since in this case the magnitude of the resultant $=P$, resolving the forces along Ax, Ay respectively, we have

$$P + Q\cos 135^\circ = P\cos\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$Q\sin 135^\circ = P\sin\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Squaring and adding (1) and (2)

$$P^2 + 2PQ\,.\cos 135^\circ + Q^2 = P^2;$$

$$\therefore\ Q + 2P\cos 135^\circ = 0;$$

$$\therefore\ Q = 2P\,.\,\frac{1}{\sqrt{2}},$$

$$\frac{Q}{P} = \frac{\sqrt{2}}{1}.$$

PROB. II. *A string ACB is fastened at its extremities to two pegs A and B, and passes through a smooth ring C. To this ring a weight W is attached, find the position of equilibrium.*

Suppose the figure to represent the system in a state of equilibrium. Then the point C is kept at rest by three forces, the weight W and the tensions of the string AC and CB respectively. And these tensions are equal to one another (each equal T suppose) since ACB is one string.

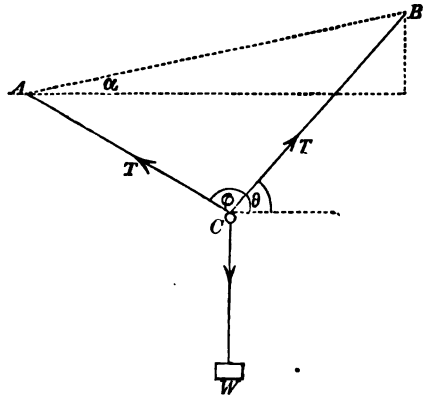
Let θ and ϕ be the angles which BC, AC make respectively with the horizon.

Then resolving the forces at C along a horizontal and vertical line, we have

$$T\cos\theta + T\cos\phi = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$T\sin\theta + T\sin\phi - W = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Here we have two equations containing three unknown quantities θ, ϕ, and T, we must therefore obtain as many more equations from the geometry of the figure as will make



the number of equations altogether as many as the number of unknown quantities involved.

Let $AB = c$, and let α be the angle AB makes with the horizon.

Let l be the length of the string, and let $CB = x$. Then from the geometry of the figure

$$c \cos \alpha = x \cos \theta - (l - x) \cos \phi \text{ ..... } (3),$$

$$c \sin \alpha = x \sin \theta - (l - x) \sin \phi \text{ ..... } (4).$$

The two statical equations (1) and (2), together with the two geometrical equations (3) and (4), give us four equations involving four unknown quantities θ, ϕ, T and x, and will therefore suffice for the solution of the problem.

From (1) $\quad \cos \theta = -\cos \phi = \cos (180 - \phi)$;

$$\therefore \theta = 180 - \phi.$$

Substituting for ϕ in (3) and (4),

$$c . \cos \alpha = x \cos \theta + (l - x) \cos \theta,$$

$$\text{i. e. } c \,.\, \cos\alpha = l\cos\theta \quad \ldots\ldots\ldots\ldots \quad (5),$$

and

$$c \,.\, \sin\alpha = x\sin\theta - (l-x)\sin\theta,$$

$$\text{i. e. } c \,.\, \sin\alpha = 2x\sin\theta - l\sin\theta \quad \ldots\ldots\ldots\ldots \quad (6).$$

From (5) we know θ, viz.

$$\theta = \cos^{-1}\left(\frac{c \,.\, \cos\alpha}{l}\right),$$

and from (6) we know x, viz.

$$x = \frac{l}{2}\left\{\frac{c\sin\alpha}{(l^2 - c^2\cos^2\alpha)^{\frac{1}{2}}} + 1\right\}.$$

Therefore the position of equilibrium is fully determined.

If we wish to find the strain on the string (T) we may easily do so from equation (2).

Notice, that when $\alpha = 0$, i. e. A and B are in the same horizontal line, $\cos\theta = \frac{c}{l}$ and $x = \frac{l}{2}$, which is evidently true.

Observation. The solution of the preceding problem is very important as illustrating the method of proceeding in solving all statical problems, which may be thus described:

(1) Draw the figure, representing the directions of all the forces by arrows.

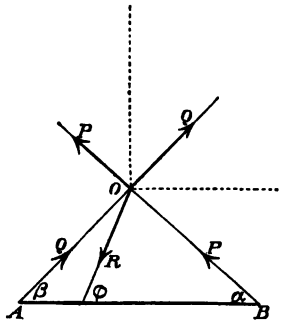
(2) Write down all the equations of equilibrium (which in the case of a particle will be two).

(3) Count the number of unknown quantities in these equations, and then add *as many more equations deduced from the geometry of the figure as will make the whole number of equations the same as the whole number of unknown quantities involved.*

The student is strongly advised to bear in mind these three rules in the solution of *every problem.*

PROB. III. *Two forces P and Q act at the extremities of a rigid rod without weight, their directions being inclined at angles α and β respectively to the rod; find what force must be applied to the beam to keep it at rest.*

Let the directions P and Q be produced until they intersect in point O. Then we may suppose the point O rigidly connected with the rod AB, by another rod without weight, and this will not alter the conditions of the problem.



The forces P and Q may therefore be supposed to act at the point O (Axiom 3, Chap. I).

Let R be the force which acting at the point O would equilibrate P and Q, ϕ the angle its direction makes with AB.

Then resolving the forces at O along and perpendicular to a line parallel to AB, we have

$$R\cos\phi + Q\cos\beta - P\cos\alpha = 0 \text{ ..... } (1),$$

$$R\sin\phi - Q\sin\beta - P\sin\alpha = 0 \text{ ..... } (2).$$

The two equations (1) and (2) will suffice to determine R and ϕ. We obtain from them

$$R^2 = P^2 + Q^2 - 2PQ\cos(\alpha+\beta),$$

and $$\tan\phi = \frac{P\sin\alpha + Q\sin\beta}{P\cos\alpha - Q\cos\beta}.$$

If G be the point at which the direction of R cuts the beam, then R may be supposed to be applied at G instead of O, and we have

$$\frac{AG}{AO} = \frac{\sin(\phi-\beta)}{\sin\phi},$$

and $\dfrac{AO}{AB} = \dfrac{\sin\alpha}{\sin(\alpha+\beta)}$;

$$\therefore\ AG = \frac{\sin\alpha \,.\, \sin(\phi-\beta)}{\sin\phi \,.\, \sin(\alpha+\beta)} \,.\, a \text{ (if } a \text{ be the length of the rod).}$$

Hence the point G is known, since ϕ has been already determined.

And a force whose magnitude is R, making an angle ϕ with the rod at the point G, will be the one required.

Examination on Chapter III.

1. Prove the parallelogram of forces as far as regards the direction of the resultant.

2. Assuming the parallelogram of forces as far as regards the direction of the resultant, shew that it holds true for the magnitude.

3. Enunciate the triangle of forces.

4. If any number of forces acting upon a particle keep it at rest, and a series of lines be drawn parallel in direction and proportional in magnitude to these forces respectively, the lines so drawn will form a polygon.

5. The sides of a quadrilateral taken in order represent in direction and magnitude four forces acting upon a particle. Shew geometrically that the sum of the resolved parts of these forces along any two straight lines at right angles are respectively zero.

6. The magnitude of two forces acting upon a particle are in the ratio of 2 to 3; find the angle between their directions when the magnitude of the resultant is a mean proportional between those of the forces. Ans. 125°. 41′.

7. In figure of Problem II. Art. 9, suppose ACW is one string passing through the ring C which is tied to the end of the other string BC, and supporting the weight W, find the strain on the peg B.

8. B is a smooth ring and A a fixed peg in the same horizontal line; a string WBA, supporting the weight W at one extremity, is passed through B and tied to A. At a point C between B and A another string is tied to the first supporting the weight P.

Find the relation between P and W in order that the vertical through C may bisect the line AB when the system is in equilibrium; supposing $AC = \frac{1}{2} AB$.

CHAPTER IV.

ON THE COMPOSITION AND RESOLUTION OF FORCES ACTING ON A RIGID BODY.

1. HITHERTO we have confined our attention to the case of forces acting upon a particle perfectly free to move; we now proceed to extend our reasoning to cases where the particle forms one of an assemblage of similar particles united together by their mutual attractions. When the number of such particles is infinite the assemblage is called a *body*. Our inquiries will be confined to that class of bodies known as *rigid*.

DEF. *A rigid body is one in which the relative position of the particles cannot be altered by the action of any finite force.*

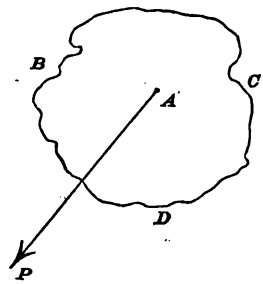
There is in fact no such thing as a rigid body in nature. Some are more nearly rigid than others. Iron for instance is more nearly rigid than wood, and wood than water. It may at first appear useless to make investigations upon a hypothesis which can never correspond with reality. Our method of proceeding is, however, to base our calculations upon this hypothesis of perfect rigidity, and afterwards to make such allowance in our results, for the want of this rigidity, as experiment shows to be necessary for the particular substance with which we happen to be dealing. In this way our final result though not mathematically exact, is yet, in most cases, sufficiently near the truth for all practical purposes.

This method of proceeding is forced upon us by our ignorance of those forces or mutual attractions which hold together the particles or molecules of a solid body. We cannot intro-

duce those molecular forces, therefore, into our conditions of equilibrium.

The principle which enables us to ignore the molecular forces is that of the "superposition of forces," enunciated in Axiom 5 (Chap. I. Art. 6). In a rigid body the molecular forces are always in equilibrium, we may, therefore, investigate the conditions of equilibrium of any system of external forces acting on a body without considering the molecular forces, knowing, that if this system of external forces be in equilibrium, the whole will be in equilibrium.

A single particle is of insensible weight, but when an infinite number are united together, forming a body, the weight of that body becomes, in many cases, a most important element in our calculation. Still the student must remember that neither the existence of this weight, nor of the molecular forces, can in any way affect the magnitude and direction of any external force acting on the body. Thus, suppose A be a particle perfectly free to move, acted upon by a force P in direction AP. The tendency of P is to make A move in the direction AP. Now suppose A be a particle of the body BDC, and acted on by the same force. The tendency of P is the same as before. Very probably, the body will not begin to move in that direction, because the action of P is modified by the weight and shape of the body. Still the magnitude and direction of the force P itself remains unaltered.



We may here also remind the student that the principle of transmission of force through a rigid body has been given in Axiom 3 (Chap. I. Art. 6).

2. The first proposition connected with this part of our subject will be "to find the resultant of any two forces acting in the same plane upon a rigid body." Such a pair of forces may be parallel or not parallel, but since by Axiom 3, any force acting upon a rigid body may be supposed to be transferred to any point in its direction rigidly connected with

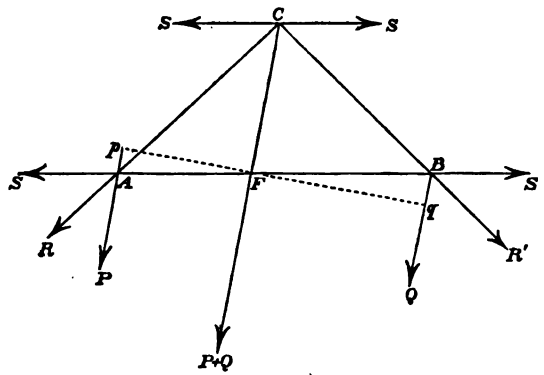
the system; and since the directions of any two forces acting upon a body and on the same plane, unless they are parallel, must intersect, we may suppose such forces to be transferred to the point of intersection of their directions, and find their resultant, by the method pointed out in the preceding chapter.

We, therefore, proceed now to the case of two *parallel* forces.

This proposition is only second in importance to that of the parallelogram of forces. The "principle of the Lever" deduced from it, is the property which enables us to extend the science of Statics from the case of forces acting on a particle to that of forces acting on a rigid body.

PROP. *To find the magnitude and direction of the resultant of two parallel forces acting on a rigid body.*

(1) Let P and Q be the forces acting at points A and B of the rigid body respectively. Join AB. Suppose for a moment



the body to be absent and the forces to act at the extremities of a rigid rod, without weight.

At A and B let two forces each equal to S be supposed to act in opposite directions along AB, these forces will manifestly not affect the equilibrium of the system.

Now S and P are two forces acting upon the particle A.

Their resultant R, acting in direction AR suppose, may therefore be obtained by the parallelogram of forces.

Similarly the resultant R' of Q and S acting at B may be obtained.

Let the direction of R and R' be produced and meet at C suppose (for it is evident they cannot be parallel), and let them be supposed to act at that point.

At C resolve them again in their original directions.

Then we shall have at C the four forces which before were acting at A and B,

viz. two equal and opposite forces acting parallel to AB, each equal to S.

And a force P and a force Q both acting along a line CF parallel to the original directions of P and Q, and cutting AB in F, suppose.

And the pair of equal forces S and S are in equilibrium.

Therefore the remaining force acting along the line CF is the resultant sought.

Its *magnitude* is $P+Q$.

Its *direction* will be known if the situation of the point F is known.

This may be determined in the following manner:

Let $AB=a$, $AF=x$.

If a force equal to R were applied at A in the opposite direction, this force and S and P would be in equilibrium, and the sides of the triangle ACF, taken in order, represent the directions of three such forces, and are, therefore, proportional to them;

$$\therefore \frac{P}{S} = \frac{CF}{x} \text{ ..... } (1).$$

Similarly,

$$\frac{Q}{S} = \frac{CF}{a-x} \text{ ..... } (2).$$

Dividing (1) by (2),

$$\frac{P}{Q} = \frac{a-x}{x};$$

$$\therefore \frac{x}{a} = \frac{Q}{P+Q}.$$

Therefore the resultant of the two parallel forces P and Q is a force $(P+Q)$ acting parallel to them at a point in the rod distant $\frac{Q}{P+Q}.a$ from A.

(2) If the forces P and Q act in opposite directions and P be greater than Q, it may be shewn in the same way that the magnitude of the resultant is $P-Q$, and that the point F through which its direction passes is such that (if $AF=x$)

$$x = \frac{Q}{P-Q}.a.$$

It will be noticed that in both the preceding cases the expression for x is independent of the angle which the direction of the forces makes with the line joining the points of application; i.e. F will remain in the same position however the direction of the parallel forces may be varied. From this property F is called the *centre* of the parallel forces.

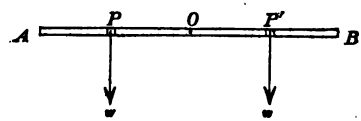
By this proposition we may find the magnitude and point of application of the resultant of a system of any number of parallel forces acting in the same direction. For we may find the resultant (R_1) of any two and its point of application, then again we may find the resultant (R_2) of R_1, and another of the forces and its point of application, and so on, until we have reduced them all to a single resultant, acting at a known point in the body. This point of application of the final resultant will have the property pointed out above, it will be the *centre* of the system of forces.

3. The weight of a body may be conceived to be the resultant of a number of parallel forces, acting at different points of the body; for the weight of any portion of matter is the force with which it is attracted towards the Earth's centre: and we may suppose any body to be divided into a number of equal and very small portions, so small that the whole weight of one such portion may be conceived to act at one point. Each one of these portions of matter will be attracted towards the Earth's centre; and since the distance from the body to the Earth's centre is very great compared with the dimensions of any body with which we shall be concerned, we may suppose all these attractions to be parallel, the resultant of this system of parallel forces is the weight of the body; and the point in the body at which this resultant acts is called the *Centre of gravity* of the body. It is evident, from what was said at the end of the last article, that the centre of gravity will remain fixed, in whatever position the body be held. For it is independent of the angle which the component attractions make with any fixed line in the body.

We shall, in a future chapter, enter more at length into the properties of the centre of gravity. The following example may serve to illustrate what we have already stated, and will be useful in some of the examples added at the end of this chapter.

Example. To find the centre of gravity of a uniform beam.

Let AB be the beam.



Take two small portions, P, P', equally distant from the centre O of the beam; the weight of each of which w may be supposed to act at the point P and P' respectively.

The resultant of these two parallel forces will be one whose magnitude is $w + w$, acting at a point between P and P' and whose distance from P (see formula Art. 2)

$$= \frac{w}{w+w} PP'$$

$$= \frac{PP'}{2}.$$

Or the resultant acts at the centre of the beam, and is equal in magnitude to the sum of the weights of the small portions P and P'.

A similar result would be produced, if we took any other pair of equal portions of matter equally distant from the centre.

And it is evident that the whole beam may be divided thus into pairs of equal portions of matter, equally distant from the centre O, and which may be taken as small as ever we please.

Hence the resultant of the weights of all the small portions into which we divide the beam, is a force equal to their sum (i. e, the weight of the whole body $= W$, suppose) acting at the centre O of the beam.

Hence the centre of the beam is the centre of gravity, and the whole weight may be supposed to act as a single force at that point.

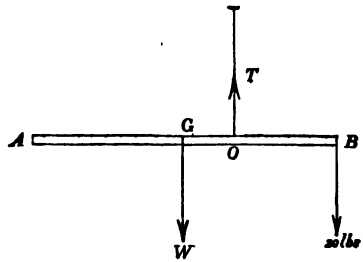
The student will be aware that all bodies are not attracted to the Earth with the same force in proportion to their bulk or volume, or, as we express it, they are not all equally heavy. A small portion of lead is as heavy as a large bulk of wool. The weight of the body depends upon its *density*. The preceding result would evidently not be true if the portion AO of the beam were made of wood, and OB of lead. When then we say the beam is uniform, we mean to imply that it is uniform not only in size from A to B, but in density also.

It may be easily deduced from the above example, that the centre of gravity of a circle or square of uniform density, is the geometrical centre of the figure.

The following is an example of the use of the formulæ for the resultant of two parallel forces.

Example. A uniform beam AB 20ft. long is suspended from a nail by a string which is fastened to the beam at a distance of 2ft. from its centre, a weight of 20lbs. is attached to the other end to keep the beam horizontal. What is the weight of the beam?

Let G be the centre of the beam.



Then we have already shewn that the whole weight of the beam (W) may be supposed to act at G.

Let T be the tension of the string, O the point in the beam to which it is attached.

Then the beam is kept at rest by the three forces W, T, and 20lbs.

Therefore T is evidently the resultant of the other two, and

$$W \cdot OG = 20 \times OB, \text{ (by Prop. page 50)},$$

but $$OG = 2, \quad \text{and} \quad OB = 8;$$

$$\therefore \; W = \frac{20 \times 8}{2} = 80\text{lbs.}$$

4. When the parallel forces are equal and act in opposite directions, the formula in Art. 2 (2) shews us that the resultant must be O in magnitude and pass through a point at an infinite distance from A, which evidently means, that such a pair of forces can have no single resultant.

DEF. Two equal forces acting in parallel and opposite directions are called *a couple.*

The perpendicular distance between their directions is called *the arm of the couple.*

The product of one of the forces and the arm is called the *moment of the couple.*

In the preceding proposition, Art. 2, we have supposed the forces to act on a rigid rod without weight: but the same holds for any rigid body which has a fixed point in the same plane as the direction of the forces. For draw through the fixed point a straight line meeting the directions of the forces in A', B' respectively; then we may suppose the force to act at these points, and the problem will be reduced to that of two forces acting at the extremities of a rigid rod.

5. DEF. If the number of units of magnitude in a force be multiplied by the number of units of length in the line drawn from a fixed point perpendicular to the direction of the force, the product is called *the moment of the force about that point.*

In Art. 2, the point F through which the resultant of P and Q passes is called the *fulcrum* of the lever AB. The property known as the principle of the lever is this, that, in the case of equilibrium the moments of the forces about F are equal. We may easily deduce this property from the result already obtained in the case of two parallel forces, for, in fig. Art. 2, draw Fp, Fq perpendicular to direction of P and Q respectively.

Then by similar triangles

$$\frac{Fp}{Fq}=\frac{FA}{FB}.$$

But we have already seen that

$$\frac{P}{Q}=\frac{FB}{FA};$$

$$\therefore \frac{P}{Q}=\frac{Fq}{Fp};$$

or $$P \times Fp = Q \times Fq,$$
the property required.

This property is also true when the forces are not parallel, and may be deduced in that case immediately from similar triangles. (See Question 4 in the examination paper at the end of this Chapter.)

6. The student will observe that the general problem which we now want to solve is *to find the conditions of equilibrium of any number of forces acting in the same plane on a rigid body.*

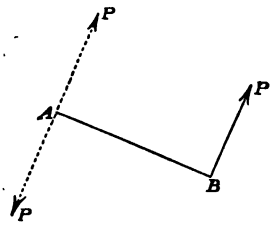
From what has been said in Article 3, it will appear that a couple is the only case of a pair of forces which cannot have a single resultant.

The student will therefore not be surprised to learn that the "theory of couples" is connected with the solution of this general problem.

The manner in which the theory of couples enters into the general problem is shewn in the following proposition.

PROP. *Any force acting on a rigid body is equivalent to the same force acting in a parallel direction at any given point in the body, together with a couple whose moment is the moment of the original force about that given point.*

Thus suppose BP to be the direction of a force P acting on a rigid body, A a given point rigidly connected with P.



At A suppose two equal and opposite forces act in a direction parallel to P and each equal to P. This will not affect the system.

Draw AB perpendicular to the direction of the original force. Then we have the force P acting at A, and a couple whose moment is $P \times AB$, instead of the original force P.

Hence if we have any number of forces acting on a rigid body in one plane, we may transfer each one to a given point, introducing a series of couples whose moments are equal to the respective forces about that point, and whose arms all have one extremity at that point.

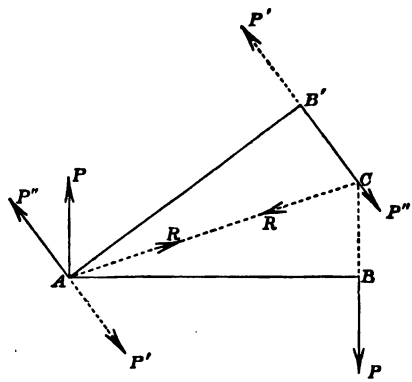
The forces acting at one point may all be reduced to a single force, and the solution of our general problem will resolve itself into finding the resultant of the series of couples which we have introduced.

The method of doing this is indicated in the two following propositions.

7. PROP. *The effect of a couple is not altered by turning its arm about one extremity in the plane in which it acts.*

Let PP be the two forces of the couple acting at an arm AB.

Through A draw AB' making any angle with AB. Make $AB' = AB$.



At B' apply two equal and opposite forces, P', P'' each equal to P, and acting at right angles to AB'.

Apply two others in the same way at A, as in the figure.

Let the directions of P at B, and P'' at B' meet at C. Join AC.

Now the triangles ABC, $AB'C$ are evidently equal in all respects.

Therefore AC bisects the angle BCB'.

And also the angle BAB'.

But since P' and P acting at C are equal, their resultant B will bisect the angle BCB', i.e. act along the line AC.

And similarly the resultant R of P' and P at A, equal to the former, will act along the line AC.

These two resultants will therefore neutralize each other, and there will only remain the two forces P'', P'', acting at the arm AB'. That is, the couple with the arm AB has been turned through the angle BAB', without altering its effect. Therefore the effect of a couple, &c. Q.E.D.

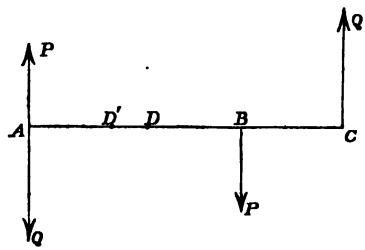
8. The preceding proposition will enable us to prove the following.

PROP. *Two couples whose arms have a common extremity and whose moments are equal have the same statical effect.*

Let the forces of one couple be each equal to P.

..... other ..... Q.

Let A be the common extremity.



Turn one couple about A until its arm coincides with the arm of the other. This, by the preceding proposition, will not alter the effect of either.

Let then AB, AC be the arms of the couples respectively, as in the figure.

Then since the moments of the couples are equal

$$P.AB = Q.AC.$$

Now P at A and Q at C, two parallel forces, are equivalent to a force $P+Q$ acting in a direction parallel to that of P and Q at a point D such that

$$AD = \frac{Q.AC}{P+Q}.$$

And, similarly, P at D and Q at A, are equivalent to a force $P+Q$ acting in the same direction at a point D', such that

$$AD' = \frac{Q.AC}{P+Q};$$

$$\therefore AD = AD';$$

therefore the two couples are reduced to two equal and opposite forces at the point D. They are therefore in equilibrium. The moments of the two couples have been here taken *opposite* in direction. It follows directly that if they be in the *same* direction, the statical effect of the two will be the same.

Therefore two couples, &c. Q.E.D.

It is evident from this proposition that if one extremity of the arm be given, the statical effect of a couple depends entirely on its moment.

The intensity of a couple is *numerically estimated* by the product of the number of units in the force and the number of units of length in the arm, which may be clearly shewn as follows.

The intensity of a couple must be measured by means of a couple chosen as the unit of couple. Let this be one of which the arm is a unit of length and the force a unit of force, such a couple will be represented by the figure 1. If another exactly similar couple be superposed on this, the effect will be doubled, and the same as if we had a single couple whose arm

was unity and force 2; and this couple being two unit-couples will be represented by the figure 2. Similarly, if n such couples were superposed, the resultant couple will be one whose arm is unity and force n, and it will be numerically estimated by n, i.e. by the numerical value of the force.

Now let the couple to be numerically estimated be one whose arm contains m units of length and whose force P units of force, whose moment, therefore, is Pm.

By the preceding proposition we may remove this couple and substitute another whose arm is unity, and whose force is such that the moment is Pm. The force of this couple is Pm, and it has already been shewn that Pm represents the intensity of such a couple, therefore Pm represents also the intensity of the couple whose force is P and arm m. That is, a couple is always *numerically estimated* by the product of the force and the length of the arm.

9. We are now in a position to solve the problem mentioned at the end of Art. 6, viz.

PROP. *To find the resultant of any number of couples, whose arms have a common extremity.*

Let P, P_1, P_2... be the forces of the couples respectively,

p, p_1, p_2... the corresponding arms.

Let all the couples be turned about the common extremity until all the arms coincide.

Now let the couple whose moment is P_1p_1 be removed and another whose arm is equal to p, and whose moment is equal to P_1p_1, be substituted for it. Let P_1' be the force of this couple.

Then
$$P_1'p = P_1p_1;$$
$$\therefore\ P_1' = \frac{P_1p_1}{p}.$$

In the same way we may substitute for P_2p_2, a couple acting at arm p, and whose force will be

$$= \frac{P_2p_2}{p},$$

and so on we may remove all the other couples until we have got instead of them a series all of whose arms coincide and equal p. It is evident that all these superimposed couples will be equivalent to a single couple acting at an arm p. If S be the force at the end of this arm we have

$$S = P + \frac{P_1 p_1}{p} + \frac{P_2 p_2}{p} + \&c.$$

$$\text{or } Sp = Pp + P_1 p_1 + P_2 p_2 + \&c.$$

i.e. the moment of the resultant couple is equal to the algebraical sum of the moments of the compound couple.

It will be noticed that there are two ways in which a couple may tend to turn a body, if we call one of these ways positive, the other must be called negative. It is usual to call these couples positive which tend to turn a body round in the direction in which the hands of a watch move, those that tend to turn the body in the opposite direction must have a negative sign prefixed to them in all algebraical expressions where they occur.

10. We are now able to solve the general problem:

To find the conditions of equilibrium of any number of forces acting upon a rigid body in one plane.

Let P, P_1, P_2, ... be the forces.

Take any point A in the body, let θ, θ_1, θ_2,..... be the angles which the directions of the forces make respectively with a fixed line passing through this point.

Let p, p_1, p_2, ... be the lengths of the perpendiculars from A on the directions of the forces.

Then, by Art. 6, we may remove P to A introducing a couple whose moment is Pp, and the extremity of whose arm is at A.

And similarly, with all the other forces.

Now let R be the resultant of all the forces at A,
ϕ the angle it makes with the fixed line.

Then by Art. 7, Chap. III.,

$$R\cos\phi = P\cos\theta + P_1\cos\theta_1 + P_2\cos\theta_2 + \ldots\ldots$$
$$R\sin\phi = P\sin\theta + P_1\sin\theta_1 + P_2\sin\theta_2 + \ldots\ldots$$

Also if M be the moment of the resultant couple we shall have also by Art. 9,

$$M = Pp + P_1p_1 + P_2p_2 \ldots\ldots$$

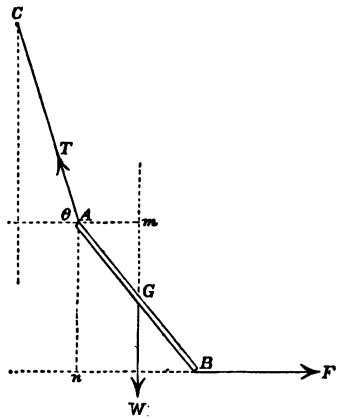
In the case of equilibrium we must have $R\cos\phi$, $R\sin\phi$ each equal to 0, as in the case of the equilibrium of a particle, and also $M=0$ (for since a *couple* cannot be counteracted by a *force* the force and the couple must separately be zero). Whence we have

$$P\cos\theta + P_1\cos\theta_1 + P_2\cos\theta_2 + \ldots\ldots = 0 \ldots\ldots\ldots \quad (1),$$
$$P\sin\theta + P_1\sin\theta_1 + P_2\sin\theta_2 + \ldots\ldots = 0 \ldots\ldots\ldots \quad (2),$$
$$Pp + P_1p_1 + P_2p_2 + \ldots\ldots\ldots\ldots\ldots\ldots = 0 \ldots\ldots\ldots \quad (3).$$

Three conditions of equilibrium for the rigid body.

11. The following problem will illustrate the use of these equations of equilibrium:

PROBLEM. A uniform beam AB, of weight W, hangs vertically from a peg C, by a string attached to the end A of the



beam, the other end of the beam is then acted upon by a horizontal force F, of such magnitude that the beam makes

an angle of 60° with the horizon. Find the strain on the string CA.

Let the figure represent the beam in the position of equilibrium. Let T be the strain or tension of the string CA. If G be the centre of the beam, the whole weight W may be supposed to act at that point (by Art. 3).

Then the beam is kept at rest by the three forces T, W, and F.

For our previous reasoning has shewn us that these three forces may be all supposed to act at one point of the body, A for instance, provided we introduce couples, whose forces shall be respectively those acting on the body, and all of whose arms shall have one extremity at A.

Draw a horizontal line through A and produce GW the direction of the weight to meet this line in m.

Draw An perpendicular to the direction of F.

Then we may conceive the beam to be kept at rest by three forces T, W, F, acting in their proper directions at A, and two couples W acting at arm Am, and F acting at arm An. And, besides the two equations of equilibrium of point A obtained in Chap. III. Art. 8, we have also the third condition that the algebraical sum of the moments of the couples about A must be nothing. Hence if θ be the angle CA makes with the horizontal line through A,

$$T\cos\theta - F = 0 \text{ ..... } (1),$$

$$T\sin\theta - W = 0 \text{ ..... } (2),$$

$$W.Am - F.An = 0 \text{ ..... } (3).$$

These are the three equations of equilibrium.

Since the angle $GAm = 60$, if $2a$ be the length of the beam, we have

$$Am = a\cos 60^\circ = \frac{a}{2},$$

and

$$An = 2a\sin 60^\circ = a\sqrt{3},$$

and equation (3) becomes,

$$W. - 2\sqrt{3}.F = 0 \ldots\ldots\ldots\ldots\ldots (4).$$

The equations (1), (2), and (4) contain three unknown quantities F, θ, and T, and will therefore be sufficient for the solution of the problem without the aid of any geometrical equations. In this case we have to find T.

From (1) and (2);

$$T^2 = F^2 + W^2.$$

From (4) $$F^2 = \frac{W^2}{12};$$

$$\therefore\ T^2 = \left(1 + \frac{1}{12}\right) W^2;$$

$$\therefore\ T = \frac{\sqrt{(13)}}{2\sqrt{3}} . W;$$

the negative sign of the radical being evidently not applicable to the case.

If it be required to find the position the beam would take when in equilibrium we have from equations (1) and (2)

$$\tan\theta = \frac{W}{F},$$

which determines θ. And the length of the string CA being known, the distance of A from the vertical through C is also known; and the position of equilibrium determined.

12. The preceding problem has been worked with a view to remind the student of the process of reasoning by which the three equations of equilibrium for a rigid body, acted on by forces in one plane, have been obtained. In practice we may lose sight of this process, and proceed at once to deduce our results from the three equations of equilibrium, as in the following problem.

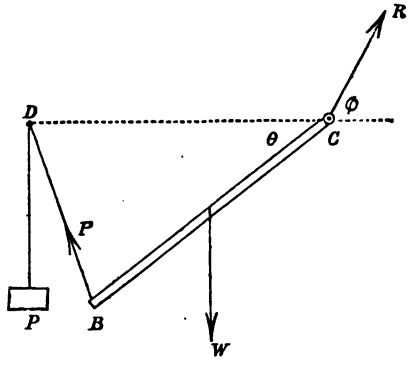
Problem. A uniform beam BC of weight W moves freely about a hinge C in a vertical plane. A string BDP is attached to the end B of the beam, and passes over the smooth pulley

D supporting the weight P at the other end, C and D are in the same horizontal line, and CB equals CD. Find the magnitude and direction of the whole force exerted on the hinge when the beam is in a position of equilibrium.

Let R be the whole force on the hinge, ϕ the angle it makes with the horizon, when the beam is in the position of equilibrium, θ the angle BCD at the same time;

$BC = CD = 2a$.

Then the beam is kept at rest by the three forces;



W acting vertically at the centre of the beam;

P the tension of the string BD;

R the action at C.

Therefore resolving horizontally and vertically, and taking moments about C, the three equations of equilibrium will be

$$R \cos \phi - P \cos CDB = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

$$R \sin \phi + P \sin CDB - W = 0 \quad \ldots\ldots\ldots\ldots \quad (2),$$

$Wa \cos \theta - P.2a \sin CBD = 0$, or since $CBD = CDB$,

$$Wa \cos \theta - P.2a \sin CDB = 0 \quad \ldots\ldots\ldots\ldots \quad (3).$$

Here we have four unknown quantities involved in these three equations,

$$R,\ \phi,\ \theta \text{ and angle } CDB;$$

we must therefore procure another relation between them before we can solve the problem; the geometry of the figure gives us

$$\text{angle } CDB = 90 - \frac{\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

which gives us such a relation as was required.

Substituting from (4) in (1), (2) and (3), we have

$$R \cos\phi = P \sin\frac{\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

$$R \sin\phi = W - P\cos\frac{\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots \quad (6),$$

$$W.\cos\theta = 2P\cos\frac{\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7),$$

from (7) we may determine θ, thus

$$W\left(2\cos^2\frac{\theta}{2} - 1\right) = 2P\cos\frac{\theta}{2};$$

$$\therefore\ \cos^2\frac{\theta}{2} - \frac{P}{W}\cos\frac{\theta}{2} = \frac{1}{2};$$

$$\therefore\ \cos\frac{\theta}{2} = \frac{P}{2W} \pm \sqrt{\left(\frac{P^2}{4W^2} + \frac{1}{2}\right)}.$$

Whence, we know $\frac{\theta}{2}$, and $\therefore\ \theta$;

dividing (6) by (5),

$$\tan\phi = \frac{W - P\cos\frac{\theta}{2}}{P.\sin\frac{\theta}{2}};$$

which determines ϕ, and substituting for ϕ and θ in (5) or (6), R also will be known.

Examination on Chapter IV.

1. Enunciate the principle of *the superposition of forces;* and explain how in investigating the conditions of equilibrium of a rigid body we are able to leave out of consideration the forces which hold together the particles of which the body is composed.

2. Two forces act in parallel directions at the extremities of a rigid rod without weight, find the position and magnitude of a third force necessary to keep the rod at rest.

Ex. Two men carry a weight between them of 100lbs. slung on a pole, the weight is 1 ft. nearer one man than the other, and the pole is 6 ft. long. How much of the weight does each man support?

3. Define the *Centre of Gravity.*

Ex. Assuming the centre of gravity of a uniform beam to be its geometrical centre; find the centre of gravity, 1st of a square, 2ndly of a circle of uniform thickness and density.

4. If two forces, not parallel, act at the extremities of a rigid rod, and O be the point where the direction of the resultant of these forces cuts the rod, shew that the moments of the forces about O are equal.

5. Define a couple. Explain the necessity of introducing the idea of a couple into mechanical investigations.

6. Explain how the intensity of a couple is estimated numerically.

7. Define the moment of a force about a point.

Ex. Two men standing on opposite sides of a vertical post pull at it by means of ropes fastened to the top of the post. Given the height of the post and the length of the ropes, compare the strength of the men when the post remains vertical.

8. A string having its extremities fixed to the ends of an uniform rod, of weight W, passes over four smooth pegs in the

same vertical plane, so as to form a regular hexagon; the rod which is horizontal being one of the sides; find the tension of the string and the vertical pressure on each peg.

9. A uniform beam PQ hangs by two strings AP, BQ from any two fixed points A and B; shew that when there is equilibrium the tensions of the strings are inversely as the sines of the angles at P and Q.

CHAPTER V.

ON THE COMPOSITION AND RESOLUTION OF FORCES ACTING ON A SYSTEM OF TWO OR MORE RIGID BODIES.

1. WHEN two bodies are in contact and at rest, it is evident that the pressure exerted by one must be exactly returned by the other.

Thus suppose a book is lying on the table, the table exerts exactly the same pressure on the book that the book does on the table. The contact, in fact, gives rise to two equal and opposite forces which equilibrate one another. The same would be true if we were to press the book down with the hand, or place another body upon it. It is usual to call the two forces arising in this way *action and reaction.* We may call which of the two we please *action*, the other will then be the *reaction.* The relation here pointed out between two such forces is expressed by saying

Action and reaction are equal and opposite.

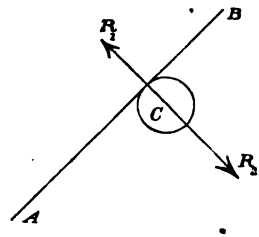
Action and reaction may arise in many other ways besides contact. For instance, if a weight be suspended by a string from a nail, the nail will exert a force equal and opposite to that of the weight. We have already, in the preceding chapters, assumed in such cases that action and reaction are equal and opposite.

We are at present concerned only with cases of contact. We will consider first the contact of two surfaces, one or both of which are smooth.

2. A smooth surface is one which cannot exert any force by contact except in a direction perpendicular to itself. It

will be seen by the very definition of a smooth surface that when two plane surfaces, one of which is smooth, are in contact, the action and reaction must act along their common normal at the point of contact. And this will also be true where one or both surfaces are curved. In every case the action and reaction will be along the common normal at the point of contact.

To make what we are saying quite plain, suppose a smooth beam to rest on a round peg C as in the figure. Here the beam will press on the peg with a certain force R_2, and the peg will press on the beam with a force R_1, and R_1 R_2 will be equal and opposite.



Further, the beam being smooth can exert no rubbing force on the peg, i. e. the action and reaction must be perpendicular to the surface of the beam, and to the surface of the circle at the point of contact.

Observe also, if we had to consider the equilibrium of the beam under certain circumstances, we might introduce the force R_1 at right angles to the beam at the point of contact, and then leave the peg out of consideration altogether.

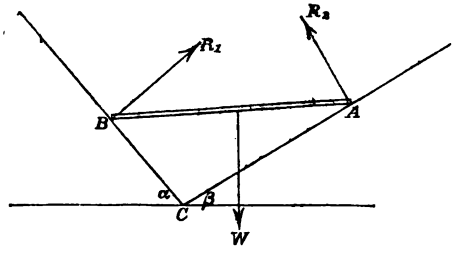
3. These considerations enable us to extend the investigation of the conditions of equilibrium of a rigid body to a system of two or more rigid bodies in contact in cases where there is no friction between the surfaces in contact. The following examples will illustrate the method.

Problem I.

A uniform beam AB lies in a vertical plane between two smooth inclined planes, CB and CA, find the position of equilibrium.

Let W be the weight of the beam collected at its middle point, R_1, R_2 the reactions against the planes CB, CA respectively.

Here we may treat the beam as though it were acted upon by the three forces R_1, R_2 and W, and disconnected from every other body.



Let α, β be the inclinations of the planes to the horizon respectively,

θ be the inclination of the beam to the horizon,

$2a$ the length of the beam.

Then, resolving horizontally and vertically and taking moments about B;

$$R_1 \sin\alpha - R_2 \sin\beta = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$R_1 \cos\alpha + R_2 \cos\beta - W = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$W.a.\cos\theta - R_2\, 2a \sin\{90 - (\beta - \theta)\} = 0 \quad\ldots\ldots\ldots (3);$$

from (1) and (2),

$$R_2 \cos\alpha . \sin\beta + R_2 \cos\beta \sin\alpha - W \sin\alpha = 0;$$

$$\text{i.e. } R_2 = \frac{W \sin\alpha}{\sin(\alpha+\beta)};$$

from (3),

$$W \cos\theta = 2R_2 \cos(\beta - \theta);$$

$$\therefore\ W = 2R_2 \cos\beta + 2R_2 \sin\beta \tan\theta;$$

$$\therefore\ \tan\theta = \frac{W - 2R_2 \cos\beta}{2R_2 \sin\beta}.$$

Substituting for R_2,

$$\tan\theta = \frac{\sin(\alpha+\beta) - 2\sin\alpha\cos\beta}{2\sin\alpha\sin\beta}$$

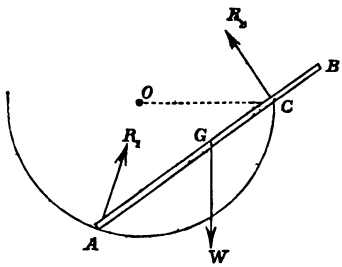
$$= \frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta},$$

which determines θ.

Problem II.

A uniform beam AB rests in a smooth semi-circular bowl, the length of the beam is twice the radius of the bowl, and the line joining the opposite lips of the bowl is horizontal. Find the position of equilibrium.

Let G be the centre of the beam, W the weight of the beam acting at G.



The reaction at A will be perpendicular to the surface of the bowl, that is, along the line AO, (O being the centre of the bowl), call this R_1.

The reaction at C will be perpendicular to the surface of the beam, call this R_2.

We have, then, to find the position of equilibrium of the beam acted upon by the three forces R_1, R_2 and W.

Let $2a$ be the length of the beam, a the radius of the circle, θ the angle the beam makes with the horizon, x the distance CG when the beam is in the position of equilibrium.

Then resolving along and perpendicular to the beam we have, observing that angle CAO = angle ACO

$$R_1 \cos\theta - W \sin\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$R_1 \sin\theta - W \cos\theta + R_2 = 0 \quad \ldots\ldots\ldots\ldots \quad (2),$$

taking moments about A,

$$W \,.\, a \cos\theta - R_2 (x + a) = 0 \quad \ldots\ldots\ldots\ldots \quad (3).$$

The statical equations (1), (2) and (3) involve the four unknown quantities θ, x, R_1 and R_2, we must therefore obtain one other relation between them from the geometry of the figure before we can solve the problem. From the triangle AOC we have

$$x + a = 2a \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

which gives us such an equation as we required.

From (3) and (4) we have, eliminating x,

$$W = 2R_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

from (1) and (2) eliminating R_1, we shall get

$$W \,.\, \sin^2\theta - W \cos^2\theta + R_2 \cos\theta = 0.$$

Substituting for R_2 from (5), we have

$$\cos^2\theta - \frac{1}{4}\cos\theta = \frac{1}{2};$$

$$\therefore \cos\theta = \frac{1}{8}\{1 \,.\, \pm \sqrt{(33)}\}\,.$$

The negative sign would give a negative value of θ, and is therefore inadmissible,

$$\theta = \cos^{-1}\left[\frac{1}{8}\{1 + \sqrt{(33)}\}\right]$$

is the required value of θ.

And from (4)

$$x = a\,(2\cos\theta - 1)$$

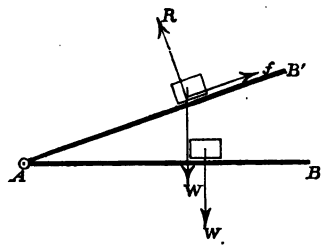
$$= a\left[\frac{1}{4}\{1 + \sqrt{(33)}\} - 1\right]$$

$$= \frac{a}{4}\{\sqrt{(33)} - 3\},$$

and the position of equilibrium is fully determined.

4. We now come to consider the case of the contact of two rough surfaces.

In this case it is evident that the mutual action of the surfaces will not be, necessarily, along the common normal at the point of contact.



Let AB be a rough plane moveable about a horizontal hinge at A, W a weight placed upon it.

When AB is horizontal the pressure of the plane is of course perpendicular to the surface in contact, whether the body be smooth or rough.

Now move AB about the hinge upwards; if the surface of W be smooth, the body will immediately slip down towards A, but if both surfaces be rough, AB may be moved into a position AB' without the equilibrium of W being disturbed.

In this case, besides the perpendicular action R, a force f will be brought into play, acting along the plane, and arising from the roughness of the surfaces in contact.

f is the force of friction.

Suppose AB' is the highest position the plane can occupy, then it is evident that for every position of the plane between AB and AB', W will be at rest; and a different amount of friction will be called into existence for every position of the plane between these limits. In every such case the friction is an unknown force to be determined by the conditions of the problem.

In one particular case, however, the friction is not unknown, but has a certain relation to the normal force R which will suffice to determine it.

This particular case is that when the body is on the point of motion. In this instance it is found by experiment that the friction varies directly as the normal pressure, and is independent of the extent of surface in contact; that is, when the friction is the greatest possible,

$$f = \mu R.$$

Where μ is independent of R, μ is called the *coefficient of friction.*

The friction being independent of the extent of surface in contact, μ will be so also; it will however vary for different kinds of surfaces.

The value of the *coefficient of friction* for different kinds of surfaces is determined by experiment, and the result set down in tables.

The following is an extract from tables constructed by M. Morin:

Surfaces in contact.	Disposition of fibres.	State of Surfaces.	Co-efficient of Friction.
Oak upon Oak	parallel	without grease	·62
		with grease .	·44
Oak upon elm	parallel	without grease	·38
Cast-iron upon cast-iron	flat	ditto	·16

The whole amount of action between the two surfaces is of course a single force, which is the resultant of the normal pressure and the friction.

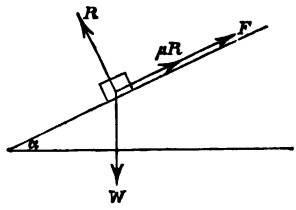
It will be noticed that the friction must always act in a direction opposite to that in which motion would begin if the bodies were smooth.

5. We now illustrate the preceding remarks with some examples.

PROBLEM I.

What force acting parallel to a plane whose inclination is α would be sufficient to keep a weight W from slipping down it; the coefficient of friction between the plane and the weight being μ?

Here if R be the normal action against the plane, and the



body be on the point of slipping down, μR will be the friction acting upwards, and if F be the required force, resolving along and perpendicular to the plane, we have

$$F + \mu R - W \sin \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$R - W \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Substituting for R in (1),

$$F = W \sin \alpha - W\mu \cos \alpha$$

$$= W(\sin \alpha - \mu \cos \alpha),$$

the required value of F.

The student will observe that if the body were on the point of moving up the plane the friction would act in the opposite direction, and if F' were the value of the force in this case,

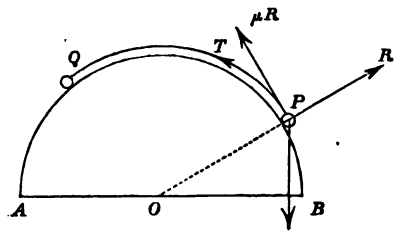
$$F' = W(\sin\alpha + \mu\cos\alpha),$$

a result which we obtain from the expression for F by simply changing the sign of μ. Any force between F and F' would keep the body at rest.

Problem II.

P and Q are two weights connected by an inextensible string and resting upon a rough semicircle. PQ subtends a right angle at the centre of the circle; find the lowest position which P can occupy consistently with equilibrium.

Let $AQPB$ be the semicircle, O its centre.



Let θ be the angle POB when P is in the stated position.

Then P is kept at rest by four forces, viz.

The pressure R acting along the normal at P.

The tension T of the string and the friction μR, both acting along the tangent at P.

And the weight of P acting vertically downwards. Resolving these forces along the normal and tangent at P respectively,

$$P\sin\theta - R = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$P\cos\theta - \mu R - T = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substituting for R from (1) in (2),

$$P\cos\theta - \mu P\sin\theta - T = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Similarly, we shall have from the two equations of equilibrium for Q,

$$Q\sin\theta + \mu Q\cos\theta - T = 0 \quad\ldots\ldots\ldots\ldots\ldots (4).$$

Eliminating T between (3) and (4),

$$(P - \mu Q)\cos\theta - (Q + \mu P)\sin\theta = 0;$$

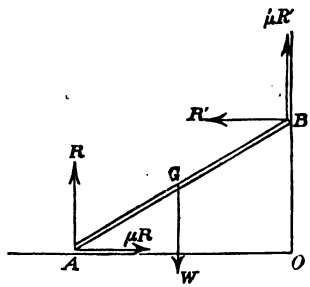
$$\therefore\ \tan\theta = \frac{P - \mu Q}{Q + \mu P},$$

which gives the required position.

Problem III.

A uniform beam rests with its lower extremity on a horizontal and its upper against a vertical plane, if μ, μ' be the coefficients of friction between the beam and the horizontal and vertical plane respectively, find the position of equilibrium, when the beam is on the point of slipping.

Let OA, OB be the horizontal and vertical walls,



R the reaction at A,

R' ..... B,

W the weight of the beam acting at the middle point G.

θ the angle BAO when the beam is in the required position of equilibrium; $2a$ the length of the beam.

Then the beam, when at the point of slipping, is kept at rest by the five forces R, μR, R', $\mu' R'$ and W as shewn in the figure.

Resolving horizontally and vertically and taking moments about A,

$$\mu R - R' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$R + \mu' R' - W = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$W . a \cos\theta - R' . 2a \sin\theta - \mu' R' \, 2a \cos\theta = 0 \quad \ldots\ldots \quad (3).$$

The equations (1), (2), (3) involve only the three unknown quantities R, R' and θ, and will therefore suffice to determine θ,

from (1) and (2),

$$R' = \frac{\mu}{1 + \mu\mu'} . W.$$

Substituting this value of R' in (3),

$$W . \cos\theta - 2 (\sin\theta + \mu' \cos\theta) \frac{\mu}{1 + \mu\mu'} . W = 0;$$

$$\therefore \; 1 + \mu\mu' - 2\mu \tan\theta - 2\mu\mu' = 0;$$

$$\therefore \; \tan\theta = \frac{1 - \mu\mu'}{2\mu}.$$

If there be no friction $\mu\mu'$ each equal 0, and $\tan\theta = \frac{1}{0}$, i.e. $\theta = 0$, the beam will only rest in a horizontal position which is manifestly true.

The student is recommended to work now as many as possible of the examples which will be found in the concluding Chapter of this book.

Examination on Chapter V.

1. Define the terms *action* and *reaction.*

2. Define *perfect smoothness.* Is such a thing as a perfectly smooth body known in practice?

3. A sphere rests on two inclined planes, shew that if α and β be the inclication of the planes respectively; R_1, R_2 the pressures; W the weight of the sphere; then

$$R_1 = \frac{W \cdot \sin \beta}{\sin (\alpha + \beta)}, \quad R_2 = \frac{W \cdot \sin \alpha}{\sin (\alpha + \beta)}.$$

4. Define *Coefficient of friction.*

5. A given body is placed upon a horizontal table which is moveable about a horizontal hinge, when the table has been elevated to make an angle α with the horizon the body begins to slip. Shew that the coefficient of friction between the body and the table $= \tan \alpha$.

6. A body is pressed against a horizontal plane in such a way that the whole pressure of the plane against the body is a force S, making an angle ϕ with the horizon. How much of this is due to friction?

CHAPTER VI.

ON THE CENTRE OF GRAVITY AND EQUILIBRIUM.

1. In Art. 3, Chap. IV. the term *centre of gravity* has been explained. It will be sufficiently evident from that explanation that every body must have a centre of gravity, and can have only one such centre; for a system of parallel forces must have one and no more than one resultant. It is also shewn in the same article that if the centre of gravity of a body be fixed in any way, the body, whatever its position, will have no tendency to turn about that fixed point.

We now proceed to discuss some other properties connected with this subject.

2. If we can by inspection determine a point around which the material of a body is symmetrically situated, that point will be the centre of gravity of the body. It has been shewn in Art. 3, Chap. IV. that the centre of a uniform beam is its centre of gravity. This result might have been arrived at by the general consideration that around the centre of the beam, the material being symmetrically situated will be equally drawn towards the earth's centre; and, therefore, the centre of the beam must be the centre of the system of forces produced by the attraction of the different parts of the beam towards the earth.

In the same way the centre of gravity of a sphere is evidently its geometrical centre, for any change in the position of the sphere can produce no change in the disposition of the material about its centre. Hence if the centre were fixed, the sphere could have no tendency to turn about that point, from one position into any other.

The centre of gravity of many geometrical figures may be determined in this way. Thus the centre of gravity of a square, or a parallelogram, is the intersection of its diagonals. The centre of gravity of an ellipse is the intersection of its axes major and minor.

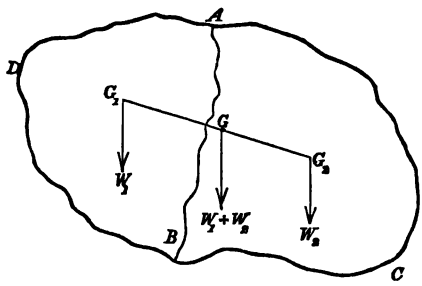
3. We may sometimes readily find the centre of gravity of a figure by the following property.

PROP. A body is divided into two parts and the centre of gravity of each part, and the weight of the whole body and one part is known: find the centre of gravity of the whole body.

Let ADC be the body,

W_1 the weight, G_1 the centre of gravity of the portion ABD,

W_2 the weight, G_2 the centre of gravity of the portion ABC.



Here then we have two parallel forces acting at G_1, G_2 respectively.

And their resultant will be a force $W_1 + W_2$ acting at a point G in the line G_1, G_2 such that

$$G_1G = \frac{W_2}{W_1 + W_2} \cdot G_1G_2,$$

(Art. 2, (1), Chap. IV.), which determines G the centre of gravity of the whole body.

COR. 1. If any two of the points G_1, G, G_2 are given, it is evident that the third may be obtained.

COR. 2. If the body were divided into n parts and G, G_1, G_2, ... G_n were the centres of gravity of the whole body, and each of the parts respectively, it is evident that n of the points G, G_1, G_2, ... G_n being given, the remaining one might be determined.

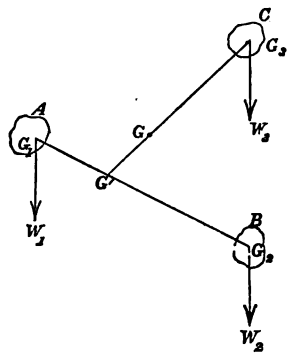
4. It is sometimes required to find the centre of gravity of a system of bodies.

PROP. To find the centre of gravity of such a system.

Let A and B be two bodies,

W_1 and W_2 their weight,

G_1 and G_2 their centres of gravity.



Then if we suppose G_1, G_2 to be connected by a rigid rod, without weight, or in some other imaginary manner, the centre of gravity will be found as in the preceding proposition, and will be a point G' in the line G_1, G_2 such that

$$G_1G' = \frac{W_2}{W_1 + W_2} \cdot G_1G_2.$$

We might suppose A and B removed and a single body whose weight was $W_1 + W_2$, and centre of gravity G' substituted in their place.

If C be a third body whose weight is W_3 and centre of gravity G_3,

The centre of gravity of the system of bodies A, B, and C will be a point G in the line joining G' and G such that

$$G'G = \frac{W_3}{W_1 + W_2 + W_3} \cdot G'G_3.$$

And similarly we might find the centre of gravity of a system of any number of separate bodies.

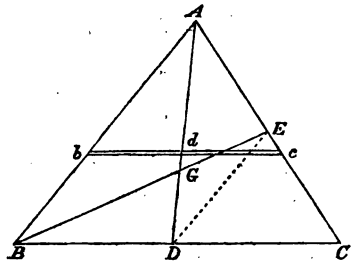
COR. 1. We might suppose the bodies A, B, C, to be different portions of the material of the same body. The preceding proposition would then, with a slight modification of reasoning, establish the fact that every system of material particles, however arranged, must have a centre of gravity.

COR. 2. It is also evident from this proposition that the centre of gravity of a body is not necessarily a point within that body.

The centre of gravity of a circle, for instance, is the centre of the circle; the centre of gravity of a hollow sphere is the centre of the sphere.

5. *To find the centre of gravity of a triangle.*

Let ABC be the triangle, (which is supposed to be of small and uniform thickness).



We may conceive this triangle to be made up of a number of rods, all parallel to BC. Let bc be one of the series of rods, the centre of gravity of bc will be d its middle point (p. 53); and d will be a point on the line AD drawn from A to the

bisection of BC. And the centre of gravity of every other one of this series of rods must also lie on this line AD. Therefore the centre of gravity of the whole triangle must be somewhere in the line AD.

Now we might have supposed the triangle to have been made up in the same way of a series of rods all parallel to the side AC, and then we should have obtained the result that the centre of gravity of the whole triangle must be somewhere in the line BE, drawn from B to the bisection of the side AC.

Since, therefore, the centre of gravity of the triangle is in the line AD, and also in the line BE, it must be the point G where those two lines intersect.

We may find the value of GD thus:

Join DE.

Then DE is parallel to BA, (Euc. VI. 2);

therefore the triangle AGB, EGD are similar;

hence

$$AG : GD = AB : ED$$

$$= 2 : 1,$$

or

$$AG = 2 . GD.$$

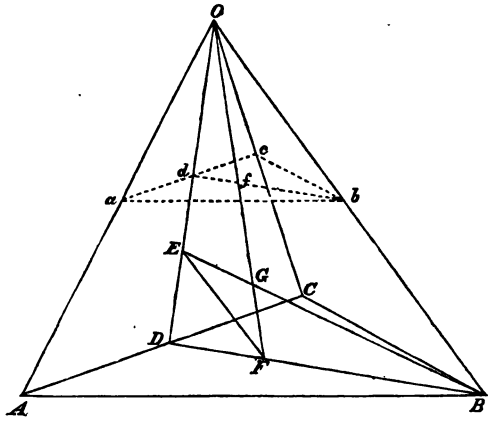
Therefore

$$GD = \frac{1}{3} AD.$$

Whence we have an easy rule for finding the centre of gravity of a triangle. *From an angle A draw a line to the bisection D of the opposite side. Take DG equal one third of the whole line DA.* G is the centre of gravity.

6. *To find the centre of gravity of a pyramid having a triangular base.*

Let $OABC$ be the pyramid, ABC being the triangular base. Bisect the edge AC in D, join OD, DB.



Take $$DE = \frac{1}{3} DO,$$

$$DF = \frac{1}{3} DB.$$

Then E and F are evidently the centres of gravity of the triangular faces AOC, ACB, respectively. Join OF, BE.

Since these two lines are in one plane, they will intersect in some point G. G is the centre of gravity of the pyramid.

For we may suppose the pyramid to be made of a number of triangles of small and uniform thickness, each parallel to the base ABC; let abc be one of these triangles.

Then, because the plane AOC meets the parallel planes ABC, abc, in the lines AC, ac, these lines are parallel.

And similarly AB, ab, are parallel, and also BC, bc.

Also if db be the line in which the plane ODB meets the plane abc, db is parallel to DB.

Let also OF meet db in f.

Then, because AC is parallel to ac;

$$\therefore\ AD : ad = OD : Od$$
$$= DC : dc,$$

and $$AD = DC; \quad \therefore\ ad = dc.$$

And similarly, since

$$DF = \frac{1}{2} BF;$$

$$\therefore\ df = \frac{1}{2} fb.$$

Therefore bd is drawn from the angle b to the bisection of the opposite side ac of the triangle abc, and f is a point in bd such that $df = \frac{1}{3} db$; therefore f is the centre of gravity of the triangle abc.

And similarly it may be shewn that every other one of this system of triangles of which we suppose the pyramid composed will have its centre of gravity in the line OF. And, therefore, the centre of gravity of the whole pyramid is somewhere in that line.

And, again, if we suppose the pyramid to be made up of triangles all parallel to the face OAC, it may be similarly shewn, that the centre of gravity of the pyramid must be somewhere in the line BE.

And hence the centre of gravity of the pyramid must be the point G, the intersection of the two lines BE and OF.

We may find GF in the following manner: join EF. Since EF cuts OD, DB, in the same proportion, it must be parallel to OB, and the triangles OGB, EGF are similar;

$$\begin{aligned} \therefore\ OG : GF &= OB : EF, \\ &= DB : DF, \\ &= 3 : 1; \end{aligned}$$

$$\therefore\ GF = \frac{1}{4} OF.$$

Hence, we have the following practical rule for finding the centre of gravity of such a pyramid:

Draw a line from one of the angles of the pyramid to the centre of gravity of the opposite face, a point in this line distant from the face one-fourth of the whole line will be the centre of gravity.

COR. 1. If the base of the pyramid be any polygon, it may easily be shewn that the centre of gravity of such a pyramid will be situated on the line joining the apex of the pyramid with the centre of gravity of the base, and will be distant from the apex $\frac{3}{4}$ths of the length of this line.

For if O be the apex of such a pyramid, G_1 the centre of gravity of the base, join OG_1. Then we may suppose the pyramid to be made up of a number of laminæ, all parallel to the base, and it can be easily shewn that the centre of gravity of each of these laminæ, and therefore the centre of gravity of the whole pyramid, is situated on the line OG_1.

Again, if we join G_1 and all the angles of the base, the pyramid will be divided into as many triangular pyramids as there are sides to the base. In OG_1 take a point G, such that $G_1G = \frac{1}{4} OG_1$. Through G draw a plane parallel to the base of the pyramid. It will be easily seen that the centre of gravity of each of the triangular pyamids, and therefore the centre of gravity of the whole pyramid, must be in this plane.

But the centre of gravity of the whole pyramid has been already shewn to be in the line OG_1.

Therefore, it must be the point G, the only point which is both in the plane and line.

Hence we may find the centre of gravity of any pyramid on a polygonal base in the manner stated above.

COR. 2. The same rule may also be extended to the case of a cone whose base is any closed curve (a circle or ellipse for instance), for such a figure may be considered

to be a pyramid whose base is a polygon of an infinite number of sides.

7. To find the centre of gravity of the *surface* of a cone on a circular base.

Such a surface may be supposed to be made up of an infinite number of isosceles triangles, each one of which has its vertex in the apex of the cone, and its base infinitely small, and coincident with the corresponding arc of the base.

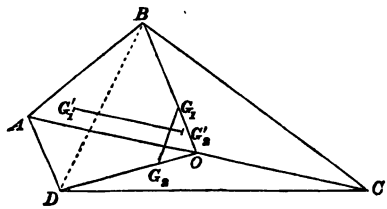
The centre of gravity of each of these triangles will be situated on the line drawn from the vertex of the triangle perpendicular to the base, and distant $\frac{1}{3}$ of the whole of that line from the base.

Hence the whole weight of the surface may be supposed to be collected in a ring, running round the surface at a distance of $\frac{1}{3}$rd of the way up the side of the cone.

The centre of gravity of such a ring is its centre, which is evidently situated in the axis of the cone, and distant from the base $\frac{1}{3}$rd of length of the axis.

8. The general formulæ for ascertaining the centre of gravity of any plane or solid figure, require a knowledge of the Integral Calculus, and, therefore, cannot be given here. Many centres of gravity may however be obtained by artifices similar to those used in the preceding articles.

It is often possible practically to obtain the centre of gravity of a plane surface very easily with a rule and compasses. For instance, suppose it be required to obtain the centre of gravity of the trapezium $ABCD$, join AC, bisect it in O.



Join BO, DO.

Take $OG_1 = \frac{1}{3} OB$. G_1 is the centre of gravity of the triangle ABC.

Take $OG_2 = \frac{1}{6} OD$. G_2 is the centre of gravity of the triangle ACD.

Therefore the centre of gravity of the trapezium is in the line G_1G_2.

Similarly, the centres of gravity G_1', G_2' of the triangles ABD, BDC may be found, and the centre of gravity of the trapezium will be in the line $G_1'G_2'$.

Therefore the centre of gravity of the trapezium is the point of intersection of the two lines G_1G_2 and $G_1'G_2'$.

The centre of gravity of any body bounded by two parallel planes may be experimentally determined by suspending it first from one point and then from another in the way pointed out in Art. 12 of this chapter.

We now proceed to give to the student two other illustrations of methods by which a centre of gravity may be sometimes found.

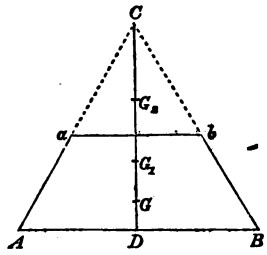
In all problems where the centre of gravity to be found is that of a plane surface, the student may verify the result by cutting out a piece of cardboard of the shape of the given body, determining the centre of gravity by the method pointed out, and then ascertaining whether the cardboard will balance upon a needle on the centre so determined.

9. ACB is an isosceles triangle from which the upper part has been cut off by a line ab bisecting the two equal sides. Find the centre of gravity of the figure $AabB$.

This may be readily done by the property pointed out in Article 3.

Let G_1, W_1 be the centre of gravity and the weight respectively of the whole triangle ACB.

G_2, W_2 be the centre of gravity and the weight respectively of the triangle aCb.



Let G be the centre of gravity of the portion $aABb$.

G_1, G_2 will both be in the line CD, drawn from C to the bisection of AB.

And, therefore, G will also be in that line.

Two parallel forces $(W_1 - W_2)$ and W_2 act at the points G, G_2, respectively, therefore their resultant W_1 acts at a point G_1, such that

$$G_1G = \frac{W_2}{W_1 - W_2} G_1G_2.$$

And W_1, W_2 are proportional to the areas of the triangles ABC, abC;

$$\therefore\ W_2 = \frac{1}{4} W_1;\ \text{also, it is evident that } G_1G_2 = \frac{1}{3} CD.$$

$$\therefore\ G_1G = \frac{1}{3} \cdot \frac{1}{3} \cdot CD = \frac{1}{9} CD.$$

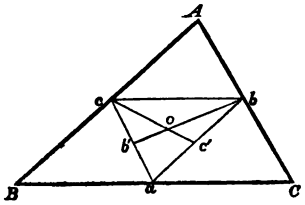
$$\therefore\ DG = \frac{2}{9} CD,$$

which determines the centre of gravity of $AabB$.

10. Three uniform heavy beams, such that their weight is proportional to their length, are joined together at their ends so as to form a triangle, find the centre of gravity of the system.

Let ABC be the triangle so formed.

Bisect the three sides in a, b, and c respectively.



Then a, b, and c will be the centres of gravity of the three beams respectively.

Hence the problem is the same as that of finding the centre of gravity of three heavy particles at a, b, and c equal to the weights of the beams BC, CA, and AB respectively.

Join ac, ab, bc.

Then, since bc cuts AB and AC in the same proportion, the two triangles ACB, Acb are similar;

$$\therefore\ BC : bc = AB : Ac$$
$$= 2 : 1;$$

i.e. bc is half BC, and similarly ab, ac are each half AB, AC.

Hence, the weights at a, b, and c being proportional to the length of the sides BC, AC, and AB, are also proportional to bc, ac, and ab respectively.

Now from b draw bb' bisecting the angle cba.

Then because bb' bisects the angle cba,

$$cb' : b'a = cb : ba$$
$$= \text{wt. of } BC : \text{wt. of } BA.$$

Therefore b' is the centre of gravity of BC and BA.

Hence we may remove the two beams and substitute a single weight at b', and the centre of gravity of the weight at

b' and the weight of AC collected at b, will be the centre of gravity of the whole three beams. But this centre of gravity must evidently be in the line bb'.

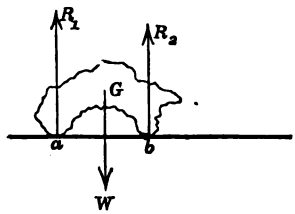
Similarly, if we draw cc' bisecting the angle bca, the centre of gravity of the three beams is in the line Cc'. Therefore its actual position is at O the intersection of these two lines bb' and cc'.

11. The centre of gravity being known we can in very many cases determine by inspection whether a body placed upon a surface will remain at rest.

If a body have one point in contact with a horizontal plane, it will remain at rest if the vertical line through the centre of gravity passes through this point of contact.

A sphere or circle, for instance, will remain at rest if placed upon a horizontal plane. For we may suppose the whole weight of such sphere or circle to be collected at its centre, the vertical line through this centre will represent the direction in which the weight acts, and will pass through the point of contact with the plane; and will there be exactly balanced by the reaction of the plane.

Again, if a body be placed so that it have two points in contact with the horizontal plane, it will remain at rest if the vertical through the centre of gravity passes through the line joining the two points of contact.

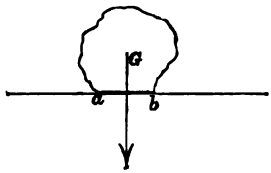


Thus, in the figure, the vertical line through G, the centre of gravity, passes through the line ab joining the two points of contact. The reactions R_1, R_2 at these two points will,

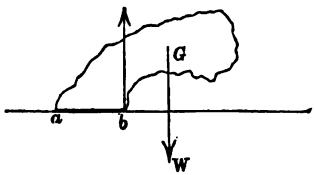
from the nature of reaction, together exactly equilibrate the weight of the body. And it is evident that the body has no tendency to turn over. It will, therefore, remain at rest.

And generally, if a body have any number of points in contact with a horizontal plane, it will remain at rest if the vertical through the centre of gravity cuts the plane at any point within the line bounding all the points of contact.

Thus, in the figure, the body has an infinite number of points, represented by the portion of its surface *ab*, in contact with the plane. It will remain at rest, since the vertical through the centre of gravity passes through the surface of contact.



The figure given below represents the effect when the body is of such a shape and so placed that the vertical through the centre of gravity does not pass through the surface in contact with the plane. The whole pressure in this case will be

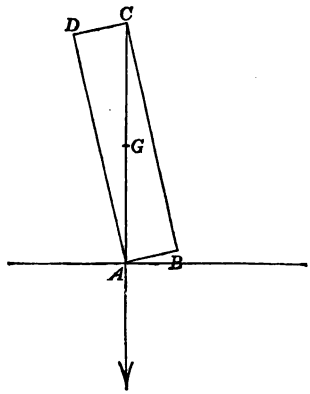


thrown on the point *b*. And the weight acting at the centre of gravity will turn the body round this point.

These considerations explain the reason why it is so difficult for a person to stand upon one leg. A person so standing must balance his body in such a way as to place his centre of

gravity vertically over the sole of his foot. When, however, he stands on both feet his centre of gravity may be anywhere, provided the vertical through it passes through the space included between the two feet. If the feet be separated by a considerable interval, the body may be placed in a great variety of postures without disturbing the person's equilibrium, or making it necessary for him to move his feet.

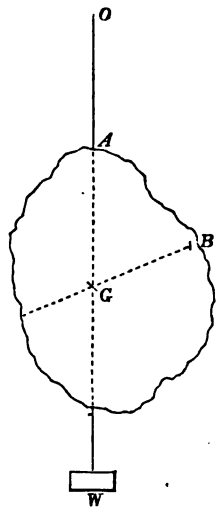
It will also be seen that a wall need not necessarily be vertical in order to stand. The walls of lofty buildings are, in fact, very seldom accurately so.



If *ABCD* be the section of a wall, (supposing the material to be sufficiently strongly put together to be considered as one mass), the wall may lean over until the vertical through the centre of gravity passes through the point *A*. If the section of the wall be a rectangle, its limiting position of equilibrium will be such that the point *D* may be the whole thickness of the wall distant from the position it would occupy if the wall were vertical.

The leaning towers of Bologna and Pisa, in Italy, are examples of this. The former is 134 feet high and $9\frac{1}{5}$ feet out of the perpendicular, and the latter 315 feet high and $12\frac{2}{5}$ feet out of the perpendicular. They do not fall, because the vertical through their centre of gravity falls within their base.

12. If a heavy body be suspended from a point it will rest with its centre of gravity in the vertical passing through the point of suspension.



For since the weight of the body acting vertically at the centre of gravity, and the tension of the string by which the body is suspended, are the only forces acting on the body, the string must evidently lie in the vertical through the centre of gravity, or these forces would not equilibrate one another.

This suggests a method by which the centre of gravity of a body may, in many cases, be experimentally determined.

Let one end of a string be fastened to the point A of the body and the other to the nail O. Let a weight W fastened to a fine string be suspended also from the nail O by the side of the body. Mark on the surface of the body the direction of the line OW. The line so marked out is the vertical through O. And the centre of gravity of the body is some point in its direction. Now let the string be detached from A and fastened to another point of the body B, and let the vertical through O be traced as before on the surface of the body. Then the point of intersection (G) of the two lines so traced will be the centre of gravity of the body.

13. Def. If a body under the action of any force be in a position of equilibrium and a *very small* displacement be given to it, if it then tend to return to the original position of equilibrium, that position is called one of *stable equilibrium.*

If, however, the body tend to move further from its original position, that position is called one of *unstable equilibrium.*

If it remain in the new position which the displacement has given it, the position is said to be *neutral.*

The following are examples of these three kinds of equilibrium:

A weight suspended by a string is an example of stable equilibrium, for if pulled slightly out of the vertical it will tend to return to its original position.

A penny balanced on its edge, or a stick balanced on the finger is a case of unstable equilibrium.

A sphere resting on a horizontal table will remain in its new position if slightly disturbed, and is therefore in neutral equilibrium.

Examination on Chapter VI.

1. Define the *Centre of Gravity* of a body; and find the centre of gravity of a uniform rigid rod.

2. Find the centre of gravity of a triangle; and deduce that of the surface of a right cone.

3. Assuming the position of the centre of gravity of a triangle, shew that that of three equal weights rigidly connected so as to form a triangle is coincident with the centre of gravity of the triangle so formed.

4. A body is divided into two parts, the centre of gravity of each of which is known, shew how to find the centre of gravity of the whole body.

Two isosceles triangles are described upon the same base; find the centre of gravity of that portion of the area of the greater which is not included by the less.

5. Four weights 1, 3, 5, 7, are rigidly connected together so as to form a square whose side is 20 inches. Find the centre of gravity of the system.

6. Explain under what circumstances a body placed on a horizontal plane will remain at rest.

What will be the corresponding condition of equilibrium when the body is placed on an inclined plane sufficiently rough to prevent slipping?

7. A triangular board rests in equilibrium with its base on a horizontal plane sufficiently rough to prevent all sliding. A force acts at its vertex parallel to the base. Find the greatest value of this force consistently with equilibrium being preserved.

8. A cube rests upon an inclined plane rough enough to prevent slipping. Find the inclination of the plane that the body may just roll down.

9. $\frac{1}{n}$th part of a triangle is cut off by a line parallel to the base; find the centre of gravity of the remaining equilateral portion.

10. Explain how the centre of gravity of a body may be experimentally determined by suspending it from a point.

11. Define the terms *stable*, *unstable*, and *neutral equilibrium*.

12. Upon a given base and with a given vertical angle which is the most *stable* triangle that can be described?

13. A wall whose centre of gravity coincides with that of its vertical section $ABCD$, is acted upon by a force F in the plane of that section. Determine the greatest value of F consistent with equilibrium.

Shew from the result what must be the weight of the wall when equilibrium is just preserved in the following case: the direction of F meets the top AB of the section at a point n such that $An = \frac{3}{10}$ ft, and $FnA = 45°$. $F = 2000$ lbs. The height AD of the wall $= 10$ ft. and the width $AB = 4$ ft.

14. A circular table rests upon a pedestal which diverges into 3 feet. The extremities of the feet touch the floor at points vertically beneath the circumference of the table. Shew that it will require a weight equal to that of the table, placed on it to upset it.

CHAPTER VII.

ON MACHINES.

1. ANY contrivance which enables us to change the point of application, direction, or intensity of a force may be called a *Machine.*

Rods for pushing, cords for pulling, surfaces and such like simple contrivances for changing the point of application or direction of a force, would, according to this definition, be machines. They are not, however, usually so reckoned.

The following six machines are for convenience regarded as simple or elementary ones:

1. The Lever.
2. The Wheel and Axle.
3. The Pulley.
4. The Inclined Plane.
5. The Screw.
6. The Wedge.

These are called the six *Mechanical Powers.*

Some writers speak of *Toothed Wheels* as a seventh mechanical power.

Every machine, however complicated, may be shewn to be composed of combinations of these so-called elementary powers.

The mechanical powers are considered, as we said, for convenience, simple or elementary machines. They are not really

so. It would be more correct to start with two, the lever and the inclined plane, as the only elementary machines, and to shew that the other four are only varieties of these; but this, though more correct, would be less convenient.

The machines with which we shall have to do are for the most part those which enable us to overcome a great resistance, or lift a great weight, by the application of a small force.

The force applied to a machine to set it in motion is called *the Power* (P), and the resistance to be overcome is termed *the Weight* (W).

Most machines are useless except when in motion. It does not fall within the province of our subject to consider them in that state. But we may determine the value of the power which would suffice to equilibrate the weight, and any increase in this value of P will of course enable us to work the machine.

The *efficiency* or *working power* of a machine will be measured by the fraction $\frac{W}{P}$.

It will be the principal object of the present chapter to find the value of $\frac{W}{P}$ in each of the mechanical powers when in a state of equilibrium.

$\frac{W}{P}$ is called the *Modulus* of a machine.

When W is greater than P the machine is said to work at a *mechanical advantage*, and when W is less than P at a *mechanical disadvantage*.

The Lever.

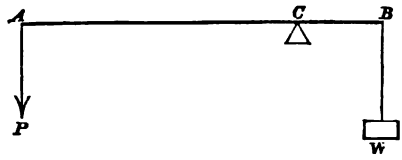
2. Def. A *Simple* or *Mathematical Lever* is a rod without weight capable of moving about a fixed point.

When the rod is not straight it is termed a *Bent Lever*. The fixed point is called the *Fulcrum*. The *power* and the *weight* are applied at two other points in the rod.

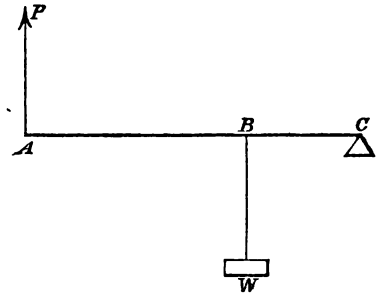
The lever is divided into three classes or orders determined by the relative position of these three points.

In the three following figures let C be the fixed point about which the lever turns, or fulcrum; A the point of application of the power (P); B the point of application of the weight (W).

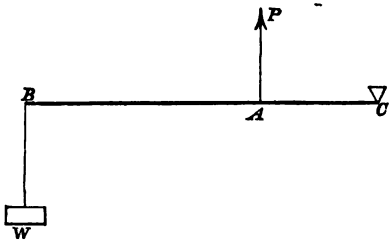
(1) In a lever of *first order* the fulcrum (C) is between the points of application of the power and weight (A) and (B).



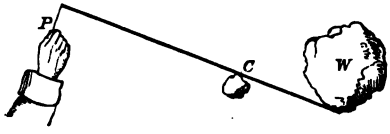
(2) In a lever of the *second order* the fulcrum is at one end, the power being at the opposite end, and the weight in the middle.



(3) In a lever of the *third order* the fulcrum is at one end and the weight at the other, the power being in the middle.



3. As an example of a lever of the *first order* we may mention *the crowbar*, used as in the figure, where *W* is a mass of rock to be raised, *C* a smaller piece of rock or a stone on which the lever is placed as a fulcrum, *P* a hand applying the

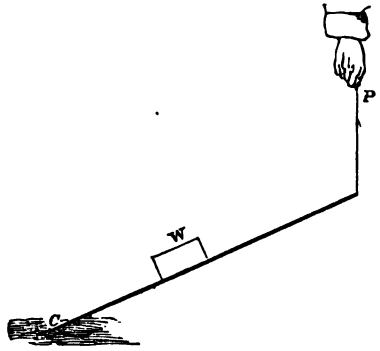


power. We know by experience that the efficiency of such a lever is increased by increasing the distance between the fulcrum and the power or by lessening the distance between the fulcrum and the weight. We shall see presently that this is *generally* true, and that *any* weight however great, might be raised (at least theoretically) by *any* power however small, by placing *C* sufficiently near to *W* and far from *P*.

Scissors and *Carpenters' pincers* are examples of pairs of levers of the first order. The power being applied by the hand, the common fulcrum being the joint, and the substance to be cut or pinched corresponding to the weight. A common *pump handle;* a *poker* between the bars of a grate raising the coals; are further examples of the same order of lever.

As an example of a lever of the *second order* we may mention *the crowbar*, when so used that its extremity rests on the

ground at a point beyond that in contact with the weight to be raised; as in the figure.



A *loaded wheelbarrow* when lifted before being put in motion is another example. The load is the weight, the fulcrum is at the wheel, and the power is applied by the man lifting.

The *oar of a boat* is also a lever of the second order. The water affords an imperfect fulcrum, the resistance of the rowlock corresponds to the weight, and the power is applied at the other extremity by hand.

Nutcrackers are a pair of levers of the same kind.

As an example of a lever of the *third order*, we have the instance of a man in the act of raising a long ladder from the ground.

Limbs of animals are also levers of this order. Take for instance the human arm, and suppose the hand to be supporting a weight. Here the elbow-joint is the fulcrum, and the power is applied by means of the muscles at a point between the joint and the weight.

4. *To find the* MODULUS *of the Lever.*

In the figures Art. 2, the directions of P and W are drawn as though they were parallel and perpendicular to the rod. This need not be the case.

Every lever, of whatever order, and whatever the directions

in which the power and weight act upon it, will be an instance of a rigid body kept at rest by three forces, viz. the power, the weight, and the reaction of the fulcrum.

Hence, by Art. 10, Chap. IV, the conditions of equilibrium will be

(1) That the forces transferred parallel to their respective directions to some point in the body, as C the fulcrum, must be in equilibrium.

(2) That the sum of the moments of the forces about the same point C must be zero.

But since C is a fixed point it is evident that the first condition may be dispensed with in this case.

And hence

if p be the length of perpendic^ar from C on the direction of P,

..w..... W,

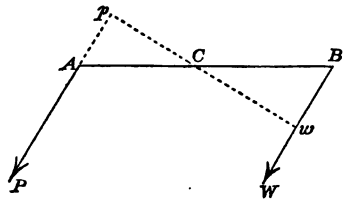
$$P \times p = W \times w,$$

is the one necessary condition of equilibrium.

And $\dfrac{W}{P} = \dfrac{\text{perpendicular from } C \text{ on direction of } P}{\text{perpendicular from } C \text{ on direction of } W}$

is the expression for the modulus or mechanical advantage of the lever.

COR. If the directions of P and W be parallel, as in the



figure, and we draw Cp, Cw perpendicular respectively to these directions, we shall have, by similar triangles,

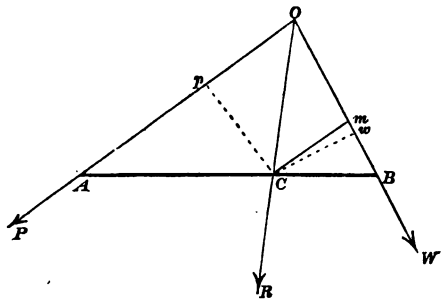
$$\frac{Cp}{Cw} = \frac{CA}{CB}.$$

Therefore $$\frac{W}{P} = \frac{CA}{CB}.$$

A property which is expressed by saying, *the power and the weight are inversely proportional to the lengths of the arms at which they act.*

We might have obtained the same property immediately from Art. 2, Chap. IV.

5. When the forces are not parallel we may obtain the expression for $\frac{W}{P}$ independently of the general equations of equilibrium for a rigid body.



For let AP, BW be the directions of the power and weight respectively, acting on the lever ACB. Let these directions be produced to meet in a point (O).

Then we may suppose P and W both to act at O, and obtain their resultant OR by the parallelogram of forces. It is evident that this resultant must pass through the fulcrum C, for a single force could under no other circumstances keep the lever at rest.

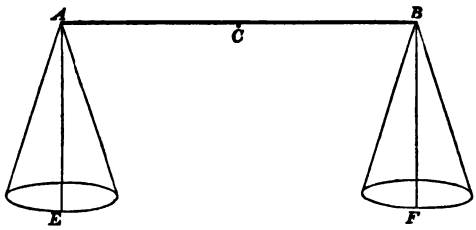
Draw Cm parallel to OP. Cp, Cw, perpendicular to OP, OW, respectively. Then the sides of the triangle COm are evidently proportional to the forces P and W, and the reaction at the fulcrum which will be equal and opposite to OR.

Therefore $\frac{W}{P} = \frac{Om}{Cm} = \frac{\sin OCm}{\sin COm} = \frac{OC.\sin COp}{OC.COm} = \frac{Cp}{Cw}$,

which is the same expression as that obtained in the previous article.

6. The different kinds of *balances*, or instruments invented for the purpose of weighing, give us some interesting examples of the practical use of the lever.

The common balance is a heavy lever of the first order, from its extremities A and B two scales E and F are suspended.



In one of these scales is placed the substance whose weight it is wished to ascertain, and in the other enough of a prepared series of weights to balance it. The scales must be of equal weight, and the arms AC, CB, of equal length and weight. Under these circumstances it will be seen that the centre of gravity of the whole balance will be a point vertically below C, and therefore, when the scales are empty, or equally loaded, the beam ACB will rest in a horizontal position.

There are three requisites to a good balance—*Horizontality*, *Stability* and *Sensibility*.

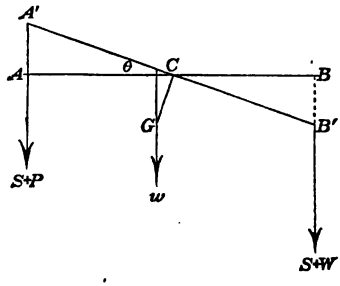
Horizontality is exhibited by the lever or beam resting in a horizontal position, when the scales are empty or equally loaded. This property may always be secured, as has been stated above, by making the strings or chains which support the scales sufficiently long to ensure the centre of gravity of the whole balance being *below* the fulcrum C.

Stability is exhibited when the balance after being disturbed quickly returns to its position of equilibrium.

Sensibility is the property which causes the balance to indicate a very small difference in the load of the scales by a large deviation of the beam from its horizontal position.

To find a measure for the sensibility of a balance.

Let AB be the beam of the balance. Let G, a point below



C, be its centre of gravity. Let S be the weight of each scale together with the tackle suspending it from A or B.

Now let P and W be two unequal weights put in the scale at A and B respectively, and let $A'B'$, making an angle θ with the horizon, be the position which the beam takes in consequence.

The lever is here kept at rest by three forces $(S+P)$ at A' $(S+W)$ at B', and the weight of the beam (w suppose) acting at G. Therefore, taking moments about C,

$$(S+P)\,A'C\cos\theta + w\,GC\sin\theta - (S+W)\,.\,B'C\cos\theta = 0\,;$$

and since C is a fixed point this is the only condition of equilibrium.

And $\quad A'C = B'C = BC;\quad \therefore$ dividing by $\cos\theta$,

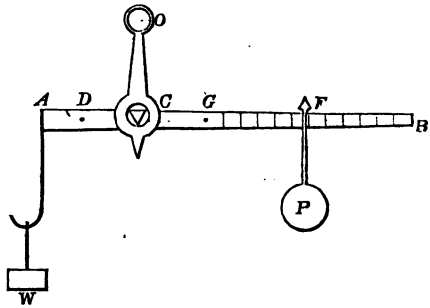
$$P\,.\,BC + w\,.\,GC\tan\theta - W\,.\,BC = 0\,;$$

$$\therefore\ \tan\theta = \frac{W-P}{w}\cdot\frac{BC}{GC}.$$

Since θ varies as $\tan\theta$ between such limits as will suffice for our present purpose, the sensibility will be greatest when $\tan\theta$ is greatest; that is, (1) when the arm (BC) of the balance is as long as possible, (2) when the weight w of the beam is as small as possible, and (3) when the distance between the centre of gravity of the beam and the point of suspension is also as small as possible.

7. The *Common Steelyard* is another form of balance.

It is represented in the annexed figure:



AB is an iron bar turning about the edge C of a triangular piece of metal, which works in an aperture of the handle by which the balance is suspended from a point O.

At A a hook or a scale is attached for the purpose of receiving any article whose weight it is wished to ascertain.

P is a moveable weight which can be suspended from any point between C and B.

Suppose W to be a body suspended from A, then P must be moved towards C or towards B, until the beam AB rests in a horizontal position.

Let F be this position of P. The bar is so *graduated* from C to B that the reading at F indicates the weight of W.

To graduate the Common Steelyard.

Let the weights W and P be removed.

Suppose G to be the centre of gravity of the beam,

W_1 ..... weight .....

The arm CG being, by construction, considerably heavier than the rest of the beam, will fall when the machine is in this unloaded state.

Let the weight P be placed on the other side of the handle CO at a point D, found by trial, such that the beam is kept in a horizontal position.

We shall then have

$$P \,.\, CD = W_1 \,.\, CG \text{ ..... } (1).$$

Now let the beam be loaded with P and W as before, then for equilibrium we must evidently have

$$P \,.\, CF + W_1 CG = W \,.\, AC.$$

Substituting from (1),

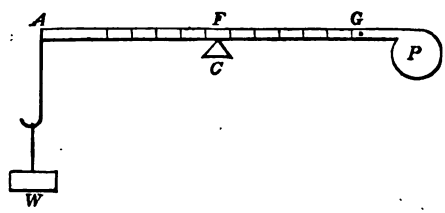
$$P(CF + CD) = W \,.\, AC;$$

$$\therefore\ W = P \,.\, \frac{DF}{AC} \text{ ..... } (2).$$

Suppose now we put for W, 1 lb, 2lbs, 3 lbs,... successively, we can calculate in each case the corresponding position of F from equation (2). At the points on the arm so determined graduations must be made and marked 1, 2, 3 ...

The space between two of these graduations may be subdivided into four or more equal parts, and these subdivisions will indicate with sufficient accuracy the alteration which must be made in W to balance P placed at any one of them. The balance in this way is so graduated that when it is loaded and P is placed, by trial, at that point F which makes the beam remain in the horizontal position, the reading at F gives the weight of W.

8. *The Danish Balance* is another form of the common steelyard.



In this case it is the fulcrum which is moveable while the counterpoise remains fixed.

The annexed figure will explain its construction.

To graduate this Balance.

Let the fulcrum C be moved about until the point G is found on which the unloaded instrument balances. G is the centre of gravity of the instrument, and its whole weight (W_1) may be supposed collected at that point. Now let the balance be loaded with the weight (W) and let F be the position of the fulcrum when the beam is horizontal in this case. The equation of moments about C will give us

$$W.AF = W_1.FG$$

$$= W_1(AG - AF);$$

$$\therefore\ AF = \frac{W_1}{W_1 + W} AG.$$

Hence making $W = 1$lb, 2lbs, 3lbs, ... successively, we shall be able to mark upon the beam the corresponding positions of the fulcrum; and dividing the space between every two such graduations and a sufficient number of equal parts, we shall, after setting the instrument, be always able to read off the weight of the substance suspended at A.

9. The following is an explanation of another balance which may be termed the *Bent Lever Balance.*

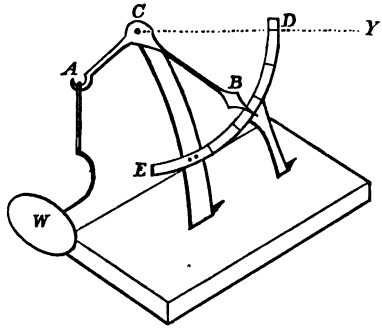
ACB is a bent lever moveable about a fulcrum C. At the end A of the arm AC a scale-pan is attached. The end B of the other arm CB is a pointer moving over a graduated arc ED. There is also a knob of metal at B to counterpoise the scale-pan.

If a weight W be placed in the scale-pan, the pointer B will move up towards D, and the reading indicated on the graduated arc will give us the weight of W.

To graduate this balance.

Let θ be the angle BCY which the arm CB makes with the horizontal. Then, the weights and centres of gravity of

the arms AC, CB, being known, and also the weight of the scale-pan, by taking moments about C we shall obtain an



expression for θ in terms of these known quantities and W. Putting, in this expression, 1 lb, 2 lbs, 3 lbs, ... successively for W, we shall find the corresponding values of θ, and may, therefore, mark on the arc the places at which the pointer will rest when the scale is so loaded. Subdividing the space between every two such divisions into a sufficient number of equal parts the balance will be fully graduated.

This method would prove in practice very inconvenient and inaccurate. The practical method of effecting the graduation is to put 1 lb, 2 lbs, 3 lbs, ... in turn in the scale-pan to mark the place at which the pointer rests.

10. The *compound lever* is a machine composed of two or more levers acting upon one another.

The following is an example of such a contrivance.

CAB is a lever of the 3rd order, united to $A'C'B'$, a lever of the 1st order, by a rod $B'A'$.

Find the value of $\frac{W}{P}$ in the case of equilibrium under the circumstances shewn in the figure.

If X be the force at B which equilibrates P, we have

$$P \times AC = X \times BC,$$

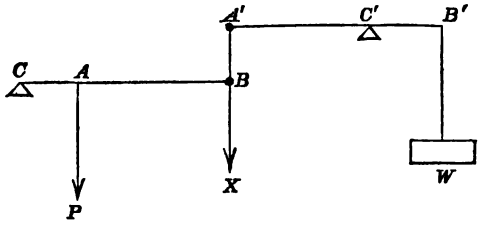
and

$$X \times A'C' = W \times B'C';$$

therefore, substituting for X,

$$P \times AC \times A'C' = W \times BC \times B'C';$$

$$\therefore \frac{W}{P} = \frac{AC \times A'C'}{BC \times B'C'}.$$

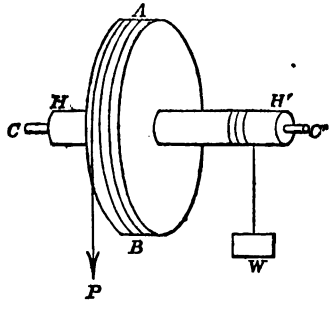


The student may now apply himself to Examination Paper No. I. at the end of this chapter.

The Wheel and Axle.

11. This machine is shewn in the figure.

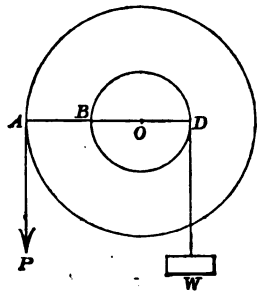
It consists of a wheel AB, and a cylinder HH', both turn-



ing round the same axis. The pivots C, C', are placed on bearings so as to allow of motion round the common axis.

The power is applied at the circumference of the wheel, often by a cord wrapped round it, as in the figure, and the weight is attached to a cord coiled round the cylinder in the contrary way, so that when P descends, W ascends, and *vice versâ.*

The figure below represents a vertical section of the machine by a plane perpendicular to the axis. The power and weight



are for simplicity's sake supposed to act in the same plane. O is the common centre. Draw through O the horizontal line $ABOD$.

A and D will evidently be the points where the cords connected with power and weight leave the wheel and cylinder respectively.

Suppose the figure to represent the machine at one moment of its action. We might for that moment suppose the whole machine reduced to a simple lever $ABOD$, of which O is the fulcrum, and A and D the points of application of the power and weight respectively. We should not by this supposition affect in any way the forces at work. Hence we see the wheel and axle is an infinite number of levers, the common fulcrum being O, the arm at which the weight acts being always equal to the radius (OD) of the cylinder, and that at which the power acts equal to the radius (OA) of the wheel. Hence the wheel and axle is sometimes called the *Perpetual Lever.*

12. *To find the modulus of the wheel and axle.*

Referring to the last figure, it will be evident from what has been already said, that in the case of equilibrium, taking moments about O, we must have

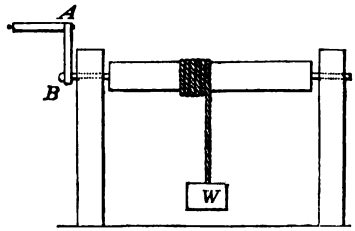
$$P \times AO = W \times OD;$$

$$\therefore \frac{W}{P} = \frac{AO}{OD} = \frac{R}{r},$$

where R is the radius of the wheel and r the radius of the cylinder. Which gives us the modulus.

13. The *Windlass* is a modification of the wheel and axle.

Here the power instead of acting at the circumference of a wheel is applied at the extremity A of an arm AB,



as in the figure. It is evident here that if the power be applied perpendicular to the arm AB, we shall have in case of equilibrium

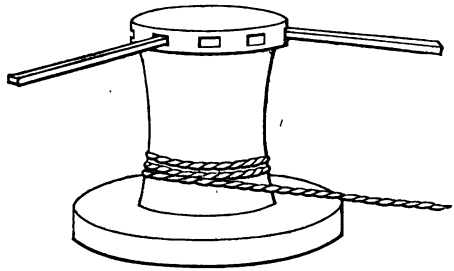
$$\frac{W}{P} = \frac{\text{length of arm } AB}{\text{radius of cylinder}}.$$

14. The *Capstan* is another example of the wheel and axle.

Here the axle or cylinder is vertical, and the power applied at the end of bars or *handspikes* by persons who travel round as the machine moves so as always to press perpendicularly against the bars.

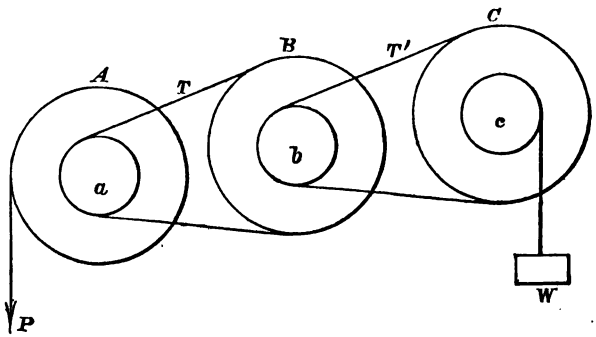
Here we have

$$\frac{W}{P} = \frac{\text{distance from the hand to the axis of the cylinder}}{\text{radius of cylinder}}.$$



15. A combination of wheels and axles may be made in the manner shewn in the figure.

A, *B*, *C*, are three wheels, *a*, *b*, *c*, the corresponding axles. The power is applied to *A*. *A* is connected with *B* by means of an endless strap passing round the axle *a* and round the



wheel *B*. This strap communicates motion by means of friction between its surface and the surface of *a* and *B*. *B* and *C* are connected in the same way and the weight *W* is attached to the axle *c*.

Now if T be the tension of the string between a and B,

..... T' ..... b and C,

taking moments about the three centres, we have, as the equation of equilibrium for each wheel and axle respectively,

$$\frac{T}{P} = \frac{\text{rad. of wheel } A}{\text{rad. of axle } a},$$

$$\frac{T'}{T} = \frac{\text{rad. of wheel } B}{\text{rad. of axle } b},$$

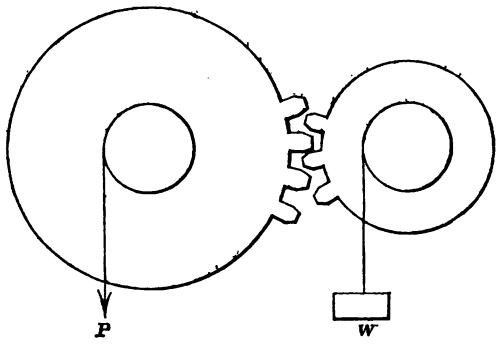
$$\frac{W}{T'} = \frac{\text{rad. of wheel } C}{\text{rad. of axle } c}.$$

Multiplying these equations together,

$$\frac{W}{P} = \frac{\text{product of radii of all the wheels}}{\text{product of radii of all the axles}},$$

which gives us the modulus of the combination.

16. It has been said that *toothed wheels* are sometimes considered as a separate mechanical power. It is better,



perhaps, to consider them as a modification of the wheel and axle.

It is evident that the combination described in the previous article would not be altered by placing the wheels as closely as possible together. Let, therefore, two wheels and axles be placed close together, and connected by teeth as in the figure; if one of them be made to revolve about its axis by a force P, motion will be communicated to the other by means of the teeth, and a weight W raised or any resistance overcome by the motion thus imparted.

Let R, r, be the radii of P's wheel and axle respectively,

.... R', r', ... W's .....

We shall have by a process similar to that of the preceding article,

$$\frac{W}{P}=\frac{R'}{R}\times\frac{r'}{r},$$

if the two axles are equal; since the teeth are set at equal distances, and their number, therefore, proportional to the radius of the wheel on which they are set; the above expression becomes

$$\frac{W}{P}=\frac{\text{number of teeth on } W\text{'s wheel}}{\text{number of teeth on } P\text{'s wheel}}.$$

The *form* of the teeth is a matter of great importance. They must be so constructed as to *roll* upon one another, and not *rub*. We have not only to determine the *best* form for this purpose, but one which will at once produce little rubbing and admit of being readily cut out in practice. The subject is very fully discussed in Willis's *Principles of Mechanism.*

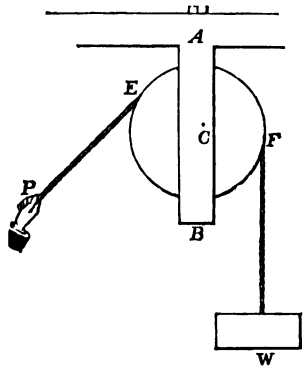
The Pulley.

17. The pulley is a wheel of wood or metal, capable of turning about an axis through its centre. It is fixed in a frame-work which is called the *block* or *sheaf.* A groove is cut in the circumference of the wheel so as to retain a rope passing over it.

When the block remains stationary the pulley is said to be *fixed*, when otherwise, it is called *moveable.*

The annexed figure represents a *single fixed pulley.*

AB is the fixed block, *C* the axis round which the wheel turns, *PEFW* the string running in the groove over the wheel.



The power is represented as applied by a hand to raise a weight *W*.

As the axis of the pulley is supposed to be quite smooth, it is evident that the tension of the string *PEFW* is the same throughout. Hence the modulus of this machine is

$$\frac{W}{P} = 1.$$

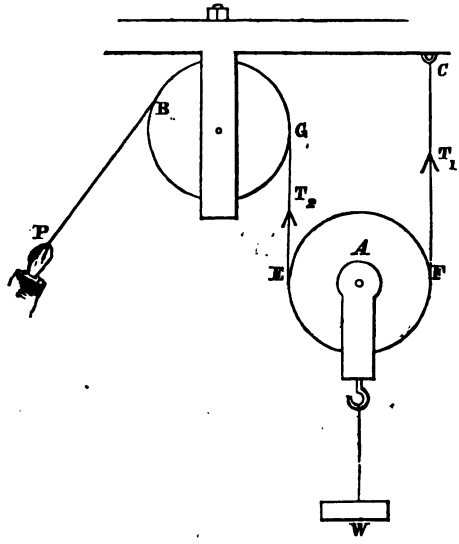
But though it gives us no mechanical advantage, the contrivance is often very convenient for changing the direction of the force; for instance, by fixing the pulley in a proper position, we may raise a weight to any height while we ourselves are standing on the ground. The effort we must exert is, however, the same which we should have to exert if we were lifting the weight without the intervention of any such contrivance.

The direction in which the power is applied will make no difference in the equation of equilibrium $P = W$; it will, however, affect the strain on the axis, for if θ be the angle be-

tween the directions EP and FW, it is easily shewn that the pressure at the axis $= 2W\cos\frac{\theta}{2}$.

18. *The single moveable pulley.*

A is a pulley, to the block of which the weight to be raised is attached.



The rope is made fast to a point C, and passes under the pulley A, and may then, for convenience, be made to pass over a fixed pulley B, as in the figure, the power being applied by the hand at P. The pulley A is called a *single moveable pulley*.

To find the modulus of this pulley.

Let F, E, be the points where the rope meets and leaves the pulley A, and G that where it meets the pulley B.

Let T_1, T_2 be the tensions of the strings CF, GE respectively.

The pulley A is kept in equilibrium by the three forces T_1, T_2 and W.

Hence, if GE be parallel to CF,

$$T_1 + T_2 = W.$$

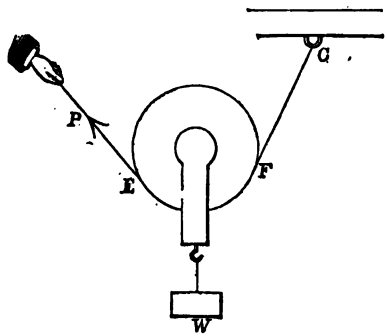
But the tension of the string $PBGEFC$ is the same throughout, therefore T_1, T_2 each equal P;

$$\therefore\ 2P = W,$$

$$\frac{W}{P} = 2.$$

Hence by this pulley we are able to raise a weight of twice the intensity of the power.

COR. If the power is not applied in such a way as to make the cord GE parallel to CF, let it act in the direction shewn in the annexed figure.



It will be evident in this case that the portions of the cord CF, PE will take a position equally inclined to the vertical.

Hence, if θ be the angle between them, resolving the forces vertically we shall have

$$2P\cos\frac{\theta}{2} = W,$$

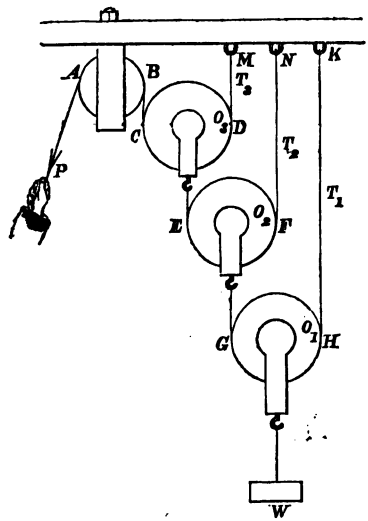
or
$$\frac{W}{P} = 2 \cos \frac{\theta}{2},$$

which is the modulus in this case.

19. *To find the modulus of a system of pulleys in which each pulley hangs by a separate string.*

This combination is generally spoken of as the first system of pulleys.

Here the last string is attached to a fixed point K, and passes under the pulley O_1, the other end being fastened to



the block of the pulley O_2, and so with the other strings as is explained by the figure. The last string passes over a fixed pulley AB.

To find the relation between the power and weight.

Let T_1, T_2, T_3 be the tensions of the strings GHK, EFN, $ABCDM$.

Then
$$2T_1 = W,$$
$$2T_2 = T_1,$$
$$2T_3 = T_2.$$

Whence multiplying and dividing by common factors, we have
$$2 \times 2 \times 2T_3 = W.$$

But
$$T_3 = P;$$
$$\therefore\ P = \frac{W}{2^3}.$$

If there were four moveable pulleys we should have had
$$P = \frac{W}{2^4};$$

and similarly if there are n moveable pulleys we should have
$$P = \frac{W}{2^n},$$
$$\text{or } \frac{W}{P} = \frac{2^n}{1}.$$

20. *To find the modulus in a system in which the same string passes round all the pulleys.*

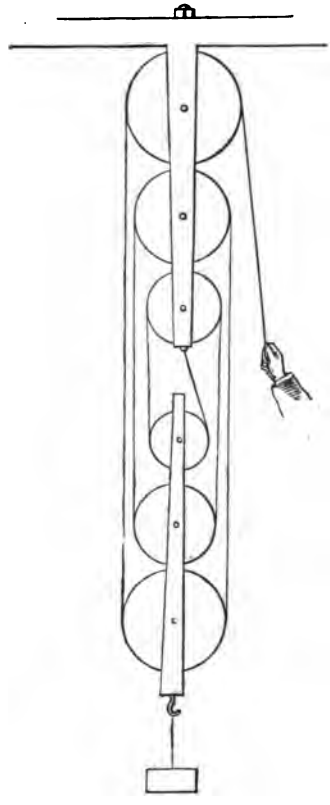
This combination is called the *Second System of Pulleys.*

Here there are two blocks, in each of which two or more wheels are fixed. The lower block is moveable, the upper one fixed. The weight is attached to the lower block, and the power applied as in the figure. The wheels of the pulleys are of such relative sizes that the different portions of the string between the pulleys are all parallel.

It is clear, in this case, that the tension of the string is the same throughout and equal to P.

Hence, if there be n pulleys altogether, the weight W will

be upheld by n tensions, and we shall have in the case of equilibrium,



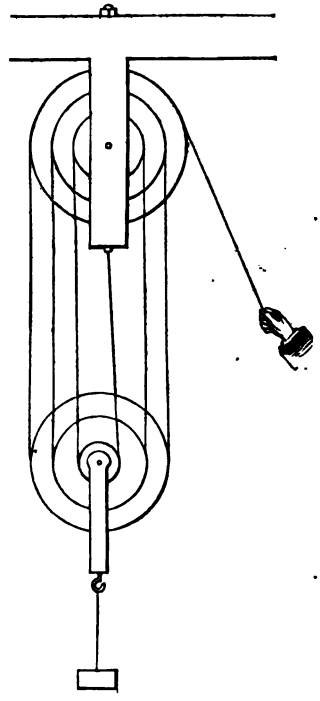
$$nP = W,$$

$$\text{or} \quad \frac{W}{P} = \frac{n}{1}.$$

Sometimes the weight of the moveable pulley is so considerable that it must be taken into account. When this is the case, if w be the weight of the lower pulleys and block, by writing $W + w$ for W in the preceding result we shall make the neces-

sary correction. And in the same way we may allow for the weight of a moveable pulley in any other combination. If there be a system of moveable pulleys we must add the weight of each to the tension of the string which supports it.

21. *White's Pulley.*



If in the system of pulleys we have described in the preceding article, we suppose, for a moment, all the pulleys equal, it will be seen, upon consideration, that when W is being raised by P, the lengths of rope which pass in the same time over the wheels of the lower block are in the arithmetical series 1, 3, 5, ... while the corresponding lengths which pass over the wheels of the upper block are in the proportion 2, 4, 6, ...

We might, therefore, so arrange the sizes of the different wheels, that the portions of rope which passed over them would make them all complete one revolution in the same time. The wheels of the upper block might then be all glued together and made to revolve as though they formed one piece on a common axis.

This is the principle of *White's pulley.* The wheels are cut out on the faces of two solid blocks, the upper one of which is fixed and the lower moveable as in the figure. The modulus of this pulley will be the same as that of the second system.

22. *To find the modulus of a system of pulleys in which all the strings are attached to the weight.*

This combination is called the *Third System of Pulleys.*

The figure will explain itself.

Let $T_1, T_2, T_3 \ldots$ be the tensions of the strings.

We shall clearly have $T_1 = P$.

The second string, whose tension is T_2, has to support two tensions each equal to P.

Hence $T_2 = 2P$.

And similarly

$$T_3 = 2T_2 = 2^2 P.$$

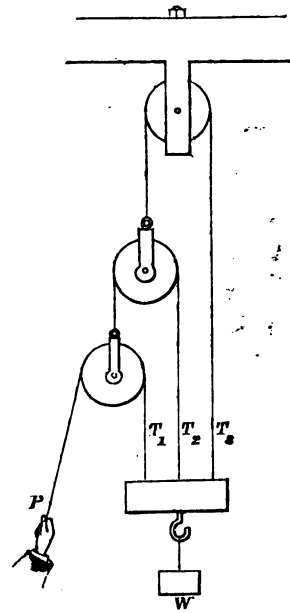
And $$T_4 = 2T_3 = 2^3 P.$$

And $$T_n = 2T_{n-1} = 2^{n-1} P.$$

Hence

$$\begin{aligned} W &= T_1 + T_2 + T_3 + \ldots + T_n \\ &= P + 2P + 2^2 P + \ldots + 2^{n-1} P \\ &= (2^n - 1) P; \end{aligned}$$

$$\therefore \frac{W}{P} = \frac{2^n - 1}{1}.$$



23. There are many more combinations of pulleys, indeed they are almost infinite in number. The student will not find any difficulty in deducing the relation between the power and the weight in the case of equilibrium, in any system that may be proposed.

A combination of pulleys is sometimes called a *Barton*, or a *Spanish Barton*.

It has already been noticed that the weights of the pulleys sometimes are so large as to make it necessary to take them into account, and a method of introducing them into our calculations has been pointed out. There are other and more important causes which affect the accuracy of our results. The *thickness of the ropes* is sometimes a very sensible quantity in proportion to the diameter of the pulley; where this is so, half the thickness of the rope must be added to the diameter of the wheel of the pulley, in all expressions where that diameter makes its appearance; and a similar precaution

must be taken in problems on the wheel and axle. Another source of error is the *friction of the axis;* a method of calculating the amount of this will be given hereafter. But the most fruitful source of inaccuracy is the rigidity of the ropes employed. When the ropes are new and the pulleys of small diameter, a considerable portion of the power is expended in bending them. Tables have been constructed for shewing what allowance must be made in our results on this account. But the practical experience of the engineer will generally be a sufficient guide on the subject.

The Inclined Plane.

24. *The Inclined Plane* is merely a plane inclined at an angle to the plane of the horizon.

The angle which it makes with the plane of the horizon is called its *inclination.*

The inclined plane is most frequently used to enable us to raise a body by a force less than the weight of the body.

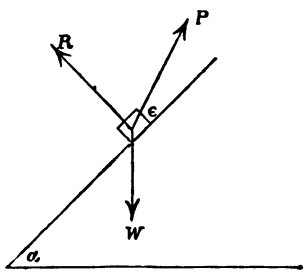
The general problem is to find what force P, acting in a given direction, will support a given weight W, resting upon a plane of given inclination; or, to find the modulus of the machine in this case.

Let α be the inclination of the plane to the horizon,

R the reaction of the plane,

W the weight of the body,

P the force acting at an angle ϵ to the horizon.



The body is evidently kept at rest by the three forces P, R, W, and resolving along and perpendicular to the plane we have

$$P \cos \epsilon - W \sin \alpha = 0 \text{ ..... } (1),$$

$$P \sin \epsilon - W \cos \alpha + R = 0 \text{ ..... } (2);$$

from (1) we have

$$\frac{W}{P} = \frac{\cos \epsilon}{\sin \alpha},$$

which gives us the modulus.

And from (2) we may find R the pressure on the plane, which will be found to be

$$= \frac{\cos (\alpha + \epsilon)}{\cos \epsilon}.$$

If P should act parallel to the plane, by putting $\epsilon = 0$ the above expression becomes

$$\frac{W}{P} = \frac{1}{\sin \alpha}.$$

If P should be horizontal we must put ϵ equal to $-\alpha$, and the expression becomes

$$\frac{W}{P} = \cot \alpha.$$

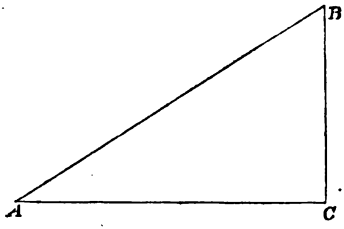
The problems connected with this machine involve no new principle; the manner of treating them has already been explained in preceding chapters.

The Screw.

25. This mechanical power is a combination of the lever and inclined plane. It consists of a cylinder with a continuous projecting thread wound round its surface. This cylinder works into a block in which there is a groove corresponding to the projecting thread; or the cylinder may have a groove and the block a projecting thread.

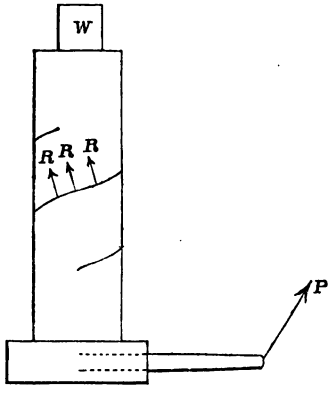
The thread or groove is of such a character as to be inclined at every point at the same angle to the axis of the cylinder.

The thread of the screw may be conceived to be generated by wrapping an inclined plane round a cylinder. Cut out a piece of paper in the shape of the right-angled triangle *ABC*, place the side *BC* on a pencil or some other cylindrical



surface so as to be parallel to the axis of the cylinder; then wrap the paper round the cylinder and *AB* will mark out a continuous descending path, which will be the thread of a screw.

The screw is generally worked by means of a bar or arm



inserted into the head of the cylinder perpendicular to its axis. One revolution of the cylinder will cause the screw to

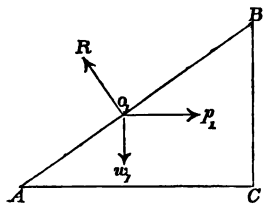
be pushed through the block or *nut* a distance equal to that between two threads of the screw.

To find the modulus in the case of the screw.

Suppose the screw to be vertical and to be used for raising a weight W, the power P being applied at the extremity of an arm, and at right angles to it, the arm and the direction of P both being in a horizontal plane, as in the figure.

The machine is evidently kept in equilibrium by the force W acting downwards, the force P, and the reaction of the thread at every point in contact with the groove represented by the forces R, R, ... in the figure.

Let now a portion of the thread be conceived to be unwrapped from the cylinder, let AB, the hypothenuse of the



right-angled triangle, ABC, be such a portion; BC being the distance between the threads, and AC the circumference of the cylinder.

Consider the equilibrium of a point O_1 on the thread.

Let R be the reaction at that point perpendicular to the thread,

w_1 the portion of W which is supported at O_1,

p_1 the portion of P which is expended at the same point.

Hence if α be the angle BAC, resolving along the line AB,

$$p_1 \cos \alpha = w_1 \sin \alpha.$$

Therefore $\frac{p_1}{w_1} = \tan \alpha$

$$= \frac{BC}{AC}$$

$$= \frac{\text{vertical distance between two threads}}{\text{circumference of cylinder}}.$$

And similarly, if $w_2 w_3 \ldots p_2 p_3 \ldots$ be the quantities corresponding to $w_1 p_1$ at points $O_2 O_3 \ldots$ respectively, we have

$$\frac{\text{vertical distance between two threads}}{\text{circumference of cylinder}} = \frac{p_1}{w_1} = \frac{p_2}{w_2} = \frac{p_3}{w_3} = \&c.$$

$$= \frac{p_1 + p_2 + p_3 + \&c.}{w_1 + w_2 + w_3 + \&c.},$$

and $w_1 + w_2 + w_3 + \&c. = W$.

And since p_1, p_2, ... all act at an arm r (if r be the radius of the cylinder), we must have, if a be the arm at which P acts,

$$(p_1 + p_2 + \&c.)\, r = P.\, a;$$

$$\therefore \frac{P}{W} = \frac{\text{vertical distance between two threads}}{\frac{a}{r}\left(\text{circumference of cylinder}\right)}$$

$$= \frac{\text{vertical distance between two threads}}{\text{circumference of circle described by } P},$$

which gives us the modulus.

The Wedge.

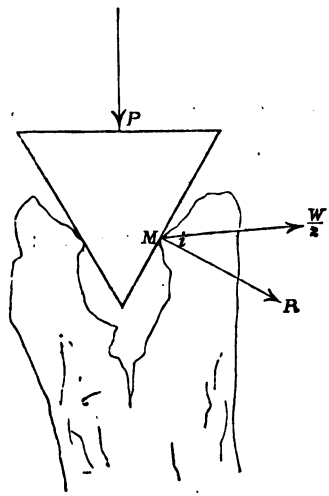
26. *The wedge* is a solid triangular prism formed of some hard material, as iron. Its most common use is to separate the parts of a body, which have from the nature of the material a tendency to approach one another. This is done by introducing the edge of the wedge between them, and then driving it forwards with the blows of a hammer.

The figure represents the section of a wedge by a plane perpendicular to its edge. This section is an isosceles triangle.

To find the modulus.

For simplicity's sake we suppose the wedge quite hard and smooth, and to be inserted between two symmetrically situated obstacles.

Let M be the point of contact of one obstacle with the wedge. Since the wedge is smooth, its action must be perpendicular to its surface. Let this normal action be R. The



action of the obstacle will take place in the direction in which M tends to move. We may call this resistance $\frac{W}{2}$ (since there will be the same resistance on the other side of the wedge, and the whole resistance will correspond to what we have in other machines called W). Let i be the angle which the direction of $\frac{W}{2}$ makes with R.

Then since the normal force R must balance the force $\frac{W}{2}$, we have

$$R \cos i = \frac{W}{2} \text{ ..... } (1).$$

Again, if P be the force with which the wedge is driven downwards, since half this force will be expanded on each side of the wedge, resolving vertically we have, if α be the angle at the edge of the wedge,

$$\frac{P}{2} = R \sin \frac{\alpha}{2} \text{ ..... } (2).$$

Substituting for R from (1),

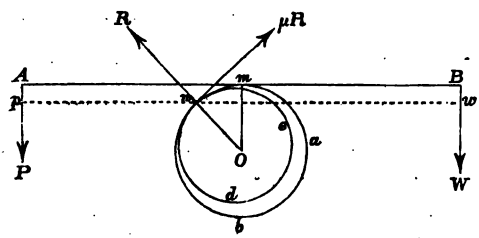
$$\frac{P}{2} = \frac{W}{2} \frac{\sin \frac{\alpha}{2}}{\cos i};$$

$$\therefore \frac{W}{P} = \frac{\cos i}{\sin \frac{\alpha}{2}},$$

which gives us the modulus.

Machines with Friction.

27. Sometimes the fulcrum of the lever is formed by



a circular hoop rigidly attached to the lever riding upon a cylindrical axis. The necessary correction on account of

friction in this case may be obtained in the following manner.

Let the figure represent a lever AB rigidly attached at the point m to the outer circle mba, and let nde be a section of the cylindrical axis.

These two circles, being nearly equal, may be considered to have a common centre O.

When AB is horizontal, the directions of P and W are supposed to be perpendicular to the length of the lever, and m to be vertically over O.

Let the figure represent the limiting position of equilibrium, i.e. the position as far as possible removed from the horizontal one consistently with equilibrium.

Let n be the point of contact of the two circles in this case. It is evident that the forces must be in equilibrium about the point n.

Draw OnR to represent the normal pressure R at n.

Draw $n\mu R$ to represent the friction μR (μ being the coefficient of friction).

Then resolving the forces along the normal and tangent at n, and taking moments about the same point we have, if angle $nOm = \theta$,

$$(P + W)\cos\theta - R = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(P + W)\sin\theta - \mu R = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and

$$W \times nw - P \times np = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where nw and np are the perpendiculars from n on the directions of P and W respectively.

But if $mB = b$, $mA = a$, and $nO = r$, we have

$$nw = b + r\sin\theta,$$

$$np = a - r\sin\theta.$$

Therefore (3) becomes

$$W(b+r.\sin\theta)-P(a-r.\sin\theta)=0 \text{ ..... } (4).$$

(1), (2), and (4) will suffice to determine the value of P in the case of equilibrium.

From (1) and (2) we have

$$\tan\theta=\mu \text{ ..... } (5),$$

which determines the point of contact.

From (4),

$$\sin\theta=\frac{Pa-W.b}{(P+W).r}.$$

From (5),

$$\sin\theta=\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}.$$

Therefore $$\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}=\frac{Pa-W.b}{(P+W).r};$$

or $$Pa=W.b+\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}.(P+W).r;$$

or $$P\left(1-\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}.\frac{r}{a}\right)=W.\frac{b}{a}+\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}.W.\frac{r}{a},$$

i.e. $$P=W.\frac{\frac{b}{a}+\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}.\frac{r}{a}}{1-\frac{\mu}{(1+\mu^2)^{\frac{1}{2}}}.\frac{r}{a}}$$

$$=W\frac{b+r.\sin\epsilon}{a-r.\sin\epsilon},$$

where for the sake of convenience we put $\mu=\tan\epsilon$. In this case ϵ is equal to θ, and therefore the result might be obtained directly from equation (4). If the friction acted in the contrary way, we could immediately find the limiting

value of P in this case by changing μ into $-\mu$, or $\sin\epsilon$ into $-\sin\epsilon$ in the above formula. Hence if P_1, P_2 be the two limiting values of P, we have

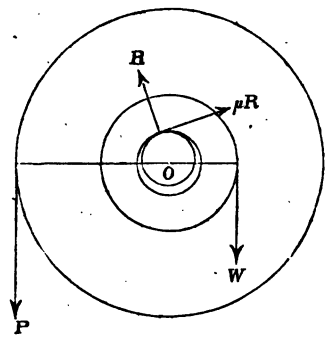
$$P_1 = W.\frac{b+r.\sin\epsilon}{a-r.\sin\epsilon},$$

$$P_2 = W.\frac{b-r.\sin\epsilon}{a+r.\sin\epsilon},$$

where $$\sin\epsilon = \frac{\mu}{(1+\mu^2)^{\frac{1}{2}}};$$

and any value of P between P_1 and P_2 will produce equilibrium.

28. In the case of the *wheel and axle* the result will be precisely the same as the above, a and b being the radii of



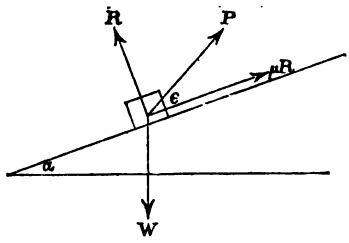
the wheel and axle respectively. The figure shews, sufficiently, the modification of the problem. The friction arising from the axis on which the *wheel of a pulley* runs will also be treated in the same way.

29. The case of the *inclined plane* with friction offers no difficulty.

Let W be a weight resting upon an inclined plane whose inclination is α.

Let it be acted upon by a force P making an angle ϵ with the plane.

Let W be on the point of slipping *down* the plane, and let P_1 be the value of P in this case; draw R the normal reaction



and the force of friction (μR) as in the figure. Resolving along and perpendicular to the plane, our equations of equilibrium are:

$$P_1 \cos \epsilon + \mu R - W \sin \alpha = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

$$R + P_1 \sin \epsilon - W \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots \quad (2).$$

Substituting in (1) the value of R from (2),

$$P_1 \cos \epsilon + \mu (W . \cos \alpha - P_1 \sin \epsilon) - W \sin \alpha = 0,$$

$$\therefore \; P_1 = \frac{W . \sin \alpha - \mu W \cos \alpha}{\cos \epsilon - \mu \sin \epsilon}.$$

If W be on the point of moving *up* the plane, and P_2 be the value of P, in this case

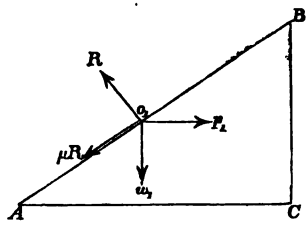
$$P_2 = \frac{W \sin \alpha + \mu W \cos \alpha}{\cos \epsilon + \mu \sin \epsilon},$$

a result which is obtained by merely changing the sign of μ.

And there will be equilibrium for any value of P between P_1 and P_2.

30. We have next to consider the effect of friction on the *screw.*

Let, in the figure, AB be a portion of the thread of the



screw unwrapped, BC being the distance between two threads, and AC the circumference of the cylinder.

If the screw be *on the point* of moving upwards, the friction will act downwards along the line AB, since AB will coincide with the tangent at the point O_1 of the thread, and using the same notation and supposing the screw to be worked in the same way as in Art. 25, resolving along and perpendicular to the line AB, we have for the equilibrium of the point O_1

$$p_1 \cos \alpha - w_1 \sin \alpha - \mu R = 0,$$

$$p_1 \sin \alpha + w_1 \cos \alpha - R = 0.$$

Whence eliminating R,

$$\frac{p_1}{w_1} = \frac{\sin \alpha + \mu \cos \alpha}{\cos \alpha - \mu \sin \alpha}.$$

And if p_2, w_2 be the quantities corresponding to p_1, w_1 for any other point in the thread O_2, we shall have exactly the same expression for $\frac{p_2}{w_2}$, and similarly for every other point.

Hence
$$\frac{\sin\alpha+\mu\cos\alpha}{\cos\alpha-\mu\sin\alpha}=\frac{p_1}{w_1}=\frac{p_2}{w_2}=\frac{p_3}{w_3}=\&c.$$
$$=\frac{p_1+p_2+p_3+\&c.}{w_1+w_2+w_3+\&c.},$$

and $w_1+w_2+w_3+\&c.=W$.

And if P_1 be the amount of the power applied at the end of the arm when the screw is just moving up, we shall have, as in Art. 25,
$$(p_1+p_2+p_3+\&c.)\,r=P_1a.$$

Hence, substituting in the above expression,
$$\frac{P_1}{W}\cdot\frac{a}{r}=\frac{\sin\alpha+\mu\,.\cos\alpha}{\cos\alpha-\mu\,.\sin\alpha},$$

or
$$P_1=\frac{r}{a}\cdot\frac{\sin\alpha+\mu\,.\cos\alpha}{\cos\alpha-\mu\,.\sin\alpha}\,.\,W.$$

And if P_2 be the value of P, when the screw is on the point of descending,
$$P_2=\frac{r}{a}\cdot\frac{\sin\alpha-\mu\,.\cos\alpha}{\cos\alpha+\mu\,.\sin\alpha}\,.\,W.$$

Any value of P between P_1 and P_2 will produce equilibrium.

31. Friction might be introduced into the expression already obtained for the equilibrium of the wedge without much difficulty, but the results obtained for this machine are from various causes of so little practical use that it seems unnecessary to dwell longer upon them.

Examination on Chapter VII.

Paper I.

1. Define the term *Machine.* Enumerate the machines known as the *six mechanical powers.* What do you mean by the *modulus* of a machine? When is a machine said to work at a *mechanical advantage?*

2. Define a simple or mathematical lever. Into what three orders are levers divided? Give an example of each order.

3. Enunciate the property which is the condition of equilibrium of the lever; and find this condition, by a geometrical method, in the case of two forces not parallel acting on a straight lever.

4. A weight of 1 cwt. is required to be raised with a straight lever of the first order, 10 ft. long, worked by a power of 8 lbs.; find the position of the fulcrum.

5. A simple lever of the first order, the arms of which are inclined to one another, without weight, has two weights, P and W, suspended from its extremities; the lever rests in such a position that the string of P makes an angle of 60°, and that of W an angle of 45° with the lever. The arms of the lever are respectively 3 and 5 feet long; find the ratio of P and W.

6. The arms of a bent lever are inclined to one another at an angle of 150°, from their extremities weights of 7 and 6 lbs. respectively are suspended. The lengths of the arms are 3 and 5 feet. Find the inclination of each arm to the horizon when there is equilibrium.

7. Describe the common balance. What is meant by the *sensibility* of the balance? Obtain an expression for measuring the sensibility, and point out what is necessary in the construction of the balance to secure this property.

8. A common balance has its arms of unequal length; a body when placed in one scale appears to weigh 4 lbs., when placed in the other, 16 lbs. What is the true weight of the body?

9. Shew how to graduate the common steelyard.

10. Shew how to graduate the Danish Balance.

11. What is a *compound* lever? Three levers of the first order are so arranged that the first applies the power to the second, and the second to the third, the powers and weights are all perpendicular to the lengths of the levers. Find the modulus.

12. A bent lever of uniform thickness, suspended by its fulcrum, rests with its shorter arm horizontal; but if the length of this arm be doubled, the other arm would be horizontal. Compare the lengths of the arms, and find their inclination.

Paper II.

1. Find the modulus of the wheel and axle. Explain why this machine has been called the *perpetual lever*.

2. If the rope used be of such thickness that it becomes necessary to take it into account, what will be the expression for the relation between P and W?

3. A weight of 1000 lbs. is sustained by a rope of 2 inches diameter; the diameter of the wheel is $1\frac{1}{2}$ feet, and that of the axle 6 inches. What must be the magnitude of the power to produce equilibrium?

4. There is a combination of wheels and axles, connected together by straps passing round an axle and the next wheel, the ratios of the radii of the wheels and axles are respectively 2 : 1, 4 : 1, 8 : 1, &c. The modulus of the machine equals m. Find the number of wheels.

5. Find the modulus of the single moveable pulley; the strings not being parallel to one another.

6. At what angle must the strings be inclined, in the single moveable pulley, in order that P may equal W?

7. Describe the arrangement of the *first system of pulleys*, and find the modulus.

How many pulleys must there be in such a system that 1 lb. may support 128 lbs.?

In the same system shew that if P and W and the weight of each pulley be all alike, there will be equilibrium.

8. Find the modulus of the second system of pulleys, and describe the principle of White's pulleys.

What must be the magnitude of P in the second system in order that it may just balance the weight of the pulleys?

9. Find the modulus of the third system of pulleys.

There are 6 pulleys in such a system; find the ratio of the weight of each pulley to W, in order that the weights of the pulleys may just support W.

10. A weight is supported on a smooth inclined plane by a force acting along the plane; shew that P is to W as the height of the plane is to the length.

11. A weight W is just supported on an inclined plane by a force P, acting by means of a wheel and axle placed at the top so that the string attached to the weight is parallel to the plane. Given the radii of the wheel and axle, find the inclination of the plane.

12. Find the modulus of the screw.

13. What weight can be sustained by means of a power of 1lb. acting on a screw, the distance between two contiguous threads of which is 1 inch, and the power being applied at

the extremity of an arm 3 yds. distant from the axis of the screw?

14. Find the modulus of the wheel and axle, taking into account the friction of the axis.

15. The inclination of a plane is known, and its roughness is such that a body will just rest upon it; find the least force, acting along the plane, necessary to draw it up.

16. The sliding weight of the common steelyard is 9 lbs. The zero-point of graduation is $\frac{1}{2}$ an inch from its fulcrum on the longer arm, and the whole beam will balance on a point 3 inches from the fulcrum on the shorter arm; what is the weight of the beam?

17. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 150 turns in the course of 12 inches, the length of the arm being 8 feet?

PROBLEMS.

In solving statical problems the student will do well to bear in mind the following rules:

1. Draw the figure, representing all the forces by straight lines and arrows, shewing the directions in which they act.

2. Write down all the equations of equilibrium. Choose the directions in which you resolve your forces, and the point about which you take your moments, so as to make your equations as simple as possible.

3. Count the unknown quantities in the statical equations, and then add as many more equations deduced from the geometry of the figure as will make the whole number of equations equal to the whole number of unknown quantities involved.

4. If the problem involve more bodies than one, the actions and reactions between them and the forces arising from friction must be represented, and the equilibrium of each body considered separately.

1. A particle whose weight is 4 lbs. is suspended freely from a point, and acted upon by a horizontal force equal to 3 lbs.; find the direction and magnitude of the force necessary to keep it from moving.

2. A smooth ring is suspended from a point. A string passes through the ring and is pulled at either end with a force of 5 lbs. in such a manner that the angle between each portion of the string so pulled and the string which suspends the ring is 30°. Find the strain upon the point of suspension.

3. The resultant of two forces represented by 1 and 2 makes an angle of 150° with the greater force. Find the magnitude of the resultant.

4. Three forces P, P, and mP keep a particle at rest; determine the angle between the equal forces.

5. A particle placed on a smooth table is pulled by three strings with forces equal to $3F$, $4F$, $6F$ respectively, and kept at rest. Find the angle between the first two strings.

6. Two forces acting at right angles have a resultant which is double the smaller force; find its direction.

7. A particle placed on a smooth table is pulled with a force of 5 lbs. by a string which makes an angle of 45° with the plane of the table. What is the force which tends to make the particle move along the table?

8. A pillar is to be pulled down by a rope of given length. At what point in the pillar must it be tied in order that the least force may be sufficient for the purpose?

9. Three forces represented by 2, 4, and 9 pounds respectively act upon a particle. Shew that they cannot produce equilibrium.

10. AB, AC, two chords of a circle, represent two forces in magnitude and direction. AB being given, find the position of AC that the resultant may be a maximum.

11. Two heavy particles are united by a string, and rest upon a smooth vertical circle; find the position of equilibrium.

12. Three smooth tacks are arranged on a vertical wall so as to form an isosceles triangle whose vertical angle is 120°, the base of the triangle being horizontal. A string with a weight (P) fastened to each end is then laid over them. Find the strain on each tack.

13. A string has one extremity fastened to a point A; it then passes through a smooth ring which supports a weight, and the other extremity is fastened to a point B in the same horizontal line with A. Given the length of the string and the distance between A and B, find the strain on A and B.

14. In the last example, suppose the line AB not horizontal. Given its length and the angle it makes with the horizon, and the length of the string, find the position of equilibrium.

15. A string is fastened at one extremity to a point A, and then passes over a smooth peg B, a weight P being attached to the other extremity. A weight W is suspended from the point C in the string $ACBP$. A and B are in the same horizontal line.

Given $AB = AC = a$, find the position of equilibrium.

16. A weight W is suspended by a string ACW from the point A. At the point C a horizontal force F acts on the string, what must be the magnitude of F that the angle ACF may be 60°.

17. Two men standing on opposite sides of a post pull at it by means of two ropes of the lengths l, l', respectively, fastened to the same point in its length. What must be the ratio of the strength of their pulls, in order that their efforts to draw the post out of the perpendicular may just neutralize one another?

18. Five forces act upon a particle. Their magnitudes are represented respectively by the figures 1, 2, 3, 4, 5, their

directions make the angles 60°, 120°, 180°, 240°, 300°, with a fixed line. Find the magnitude and direction of a force which will just counterbalance the tendency of the particle to move.

19. A weight W is sustained on a smooth inclined plane by 3 forces each equal to $\frac{W}{3}$, one acting vertically upwards, another horizontally, and a third parallel to the plane; find the inclination of the plane.

20. Two given weights P and Q rest upon two smooth inclined planes, and are attached together by means of a string passing over the common vertex of the planes. Find the ratio of the inclinations of the planes.

21. Shew that if a system of forces be represented in magnitude and direction by the sides of a polygon taken in order, the moment of the forces about any point may be represented by twice the area of the polygon.

22. A circular arc without weight rests with its convexity downwards on a horizontal table. Two weights P and Q are suspended from its two extremities. Determine the position of equilibrium.

23. Two parallel forces acting upon a rigid line, in opposite directions, are represented in magnitude by the figures 3 and 5, and their directions are 12 inches apart; find the magnitude and position of a third force necessary to keep them in equilibrium.

24. A given uniform beam rests with one end in a smooth hemispherical bowl, and the upper end against a smooth vertical wall. Find the position of equilibrium.

25. A piece of wire is formed into a triangle; find the distance of the centre of gravity from each of the sides: and shew that if x, y, z be the 3 distances, and r the radius of the inscribed circle, then

$$4xyz - r^2(x+y+z) - r^3 = 0.$$

26. The sum of the squares of the three sides of a triangle equals three times the sum of the squares of the lines drawn from the centre of gravity to the three angular points of the triangle.

27. If the centre of gravity of a four-sided figure coincide with one of its angular points, shew that the distances of this point and the opposite angular point from the line joining the other two angular points are as 1 to 2.

28. Two given heavy particles are connected together by an inflexible rod without weight and of given length, and placed in a smooth hemispherical bowl; find the position of equilibrium.

29. Two small rings slide on a smooth vertical circle; a string passes through both rings and has 3 equal weights attached to it, one at each end, and one between the rings; find the position of the rings in the case of equilibrium.

30. A uniform heavy rod of given length is to be supported in a given position with its upper end resting against a smooth vertical wall by means of a string attached to the lower end of the rod and a point in the wall. Find the point in the wall to which it must be attached.

31. AB is a rod capable of turning freely about its extremity A, which is fixed; CD is another rod equal to $2AB$, and attached at its middle point to the extremity B of the former, so as to turn freely about this point. A given force acts at C in the direction CA. Find the force which must be applied at D in order to produce equilibrium.

32. A given rod rests with its upper end against a smooth vertical wall and the lower end suspended by a given string fastened to a point in the wall. Find the position of equilibrium.

33. A uniform beam AB moveable about a hinge at A in a vertical plane leans upon a prop CD in the same vertical plane; determine the strain upon the prop.

34. A rigid rod without weight lies with one end against the smooth internal surface, and the other end projecting beyond the rim of a conical shell whose axis is vertical. A weight (W) is suspended from the projecting end of the beam; find the position of equilibrium of the rod.

35. A hemisphere is placed upon a rough inclined plane; given the coefficient of friction between the surfaces; find the limiting position of equilibrium.

36. A string fastened to a point in a rough vertical wall is wrapped round a ball which is then allowed to hang down against the wall. Determine the limiting position of equilibrium.

37. A uniform beam rests with its middle point upon a rough vertical circle; find the greatest weight that can be suspended from one end of the beam without its sliding off the circle.

38. A weight is suspended from a point in the circumference of a hoop. The hoop is hung on a rough peg, and rests so as to be on the point of slipping round on the peg. Find the position of equilibrium.

39. Two circular discs hang in a vertical plane by means of a string which passes over a smooth peg; the two ends of the string being attached to the two centres of the discs respectively. The circumferences of the discs are smooth, and are placed in contact so as to keep the system in equilibrium. Shew that the portions of the strings on either side of the peg are to one another as the squares of the radii of the discs.

40. Two equal and uniform beams AC, BC are connected by a smooth hinge at C, and are placed upright on a horizontal plane with the ends A and B resting on the plane; if β be the greatest value of the angle ACB which is consistent with equilibrium, shew that the coefficient of friction $= \frac{1}{2} \tan \frac{\beta}{2}$.

41. If a horse exert a traction of 160 lbs., what weight will he pull up a hill which has a rise of 3 in 100; supposing the coefficient of friction to be $\frac{1}{12}$?

42. Two rough bodies rest on an inclined plane and are connected by a string which is parallel to the plane. If the coefficient of friction be not the same for both, find the greatest inclination of the plane consistent with equilibrium.

43. A triangular wedge rests upon a rough horizontal table. The side which forms an inclined plane is smooth, and upon it a heavy particle is supported by a force making an angle with the inclined plane. Find the whole pressure on the horizontal table.

44. A uniform beam rests against a peg at the focus of a parabola, its lower extremity being supported by the curve. The axis of the parabola is vertical and its surface smooth; determine the inclination of the rod to the horizon.

45. In the system of pulleys where each string is attached to the weight, if one string be nailed to the block through which it passes, shew that the power may be increased up to a certain extent without producing motion. If there be three pulleys and the action of the middle one be checked in the manner described, find the tension of each string for given values of P and W.

46. If the axis about which a wheel and axle turns coincide with the axis of the axle but not with that of the wheel; find the greatest and least ratios of the power and weight necessary for equilibrium, neglecting the weight of the machine.

47. The arms of a balance are unequal, and one of the scales is loaded. A body whose true weight is P lbs. appears to weigh W lbs. when placed in one scale, and W' lbs. when placed in the other; find the ratio between the arms, and the weight with which the scale is loaded.

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